SUB-LOGARITHMIC HEEGAARD GRADIENTS.

CLAIRE RENARD.

Introduction

Thurston conjectured that every complete hyperbolic, connected and orientable 3-manifold of finite volume virtually fibers over the circle, i.e. such a manifold has a finite covering that is a bundle over the circle.

Lackenby [Lac2006] proposed a program in order to solve this conjecture. This program includes two conjectures about two new 3-manifold invariants : the infimal Heegaard gradient and the infimal strong Heegaard gradient.

Let N be a connected, compact and orientable 3-manifold. The Heegaard Euler characteristic $\chi_-^h(N)$ of N is the minimum over all Heegaard surfaces F of the negative part $\chi_-(F) = \min\{-\chi(F), 0\}$ of the Euler characteristic of F. Likewise, the strong Heegaard Euler characteristic $\chi_-^{sh}(N)$ is the minimum of $\chi_-(F)$ over all the strongly irreducible Heegaard surfaces F of N. As usual, if the manifold N does not contain any strongly irreducible Heegaard surface, $\chi_-^{sh}(N) = +\infty$. For further definitions and details about the theory of Heegaard splittings, see section 1.

Definition 1. [Lac2006, p. 319 et 320]

*Let M be a compact, connected and orientable 3-manifold. One defines the **infimal Heegaard gradient** of the manifold M as :*

$$\nabla^h(M) = \inf_{i\in I}\left\{\frac{\chi_-^h(M_i)}{d_i}\right\},$$

where the infimum is over the family of all coverings $M_i \to M$ of M with finite degree d_i.

*Likewise, the **infimal strong Heegaard gradient** of the manifold M is :*

$$\nabla^{sh}(M) = \inf_{i\in I}\left\{\frac{\chi_-^{sh}(M_i)}{d_i}\right\},$$

where $\chi_-^{sh}(M_i)$ is the strong Heegaard Euler characteristic of the finite covering $M_i \to M$.

Results of Lackenby show that those two quantities provide information about the existence of incompressible surfaces in finite covers of a manifold M with sufficiently large degrees. They led Lackenby to formulate the following conjectures.

Conjecture 1 (Heegaard gradient Conjecture)**.** [Lac2006, p. 320]

The Heegaard gradient of a compact, connected and orientable hyperbolic 3-manifold is zero if and only if the manifold M virtually fibers over the circle $\mathbb{S}^1$.

Conjecture 2 (Strong Heegaard gradient Conjecture)**.** [Lac2006, p. 320]

The strong Heegaard gradient of a closed, connected and orientable hyperbolic 3-manifold is always strictly positive.

Maher [Mah] has established the following fibration criterion. This criterion is weaker than Lackenby's conjecture, and can be obtained from Theorem 1.1 p. 2228 of [Mah].

Theorem (Maher)**.** *A closed, connected and orientable hyperbolic 3-manifold virtually fibers over the circle if and only if it has an infinite family of finite coverings $(M_i \to M)_{i\in I}$ with uniformly bounded Heegaard genus.*

Date: September 29, 2009.

The aim of this article is to provide a virtual fibration criterion standing between Lackenby's conjecture and Maher's result. We want to allow the quantity $\chi_-^h(M_i)$ to grow, but sub-logarithmically with the covering degree. To this aim, we studied Maher's proof and tried at each stage to produce explicit bounds. When some proofs needed details to become effective, we gave them. Maher himself already suggested the possibility to deduce such a result from his proof [Mah, p. 2228], but without explicit statement.

To give a precise statement, we introduce two new Heegaard gradients. Those gradients are in the spirit of Lackenby's [Lac2006], but we replaced the denominator by a sub-logarithmic function of the covering degree d_i. The main goal of this article is to use Maher's techniques to prove a sub-logarithmic version of Lackenby's conjectures 1 and 2.

Theorem 1. *Let M be a connected, orientable and closed hyperbolic 3-manifold. If there exists an infinite family $(M_i \to M)_{i\in I}$ of coverings of M with finite degrees d_i such that*

$$\inf_{i\in I} \frac{\chi_-^h(M_i)}{\sqrt{\ln\ln d_i}} = 0,$$

then for infinitely many indices i, the manifold M_i contains an embedded surface that is a virtual fiber. In particular, the manifold M virtually fibers over the circle $\mathbb{S}^1$.

In this result, the function $\ln(d_i)$ arises from a comparison between the volume and the diameter of a properly chosen submanifold of the hyperbolic manifold M_i. The square root and the first logarithm come from a combinatorial estimation.

In fact, there exists a constant k depending only on the choice of a Dirichlet fundamental domain $\mathcal{D}$ for M such that if the finite cover M_i of M satisfies $\chi_-^h(M_i)/\sqrt{\ln\ln d_i} \leq k$, then M_i contains an embedded virtual fiber. Nevertheless the constant k is quite difficult to write down explicitly.

As we do not a priori suppose that the Heegaard genus of the covering M_i is uniformly bounded, this statement is an improvement of Maher's Theorem 1.1 [Mah]. The theorem 1 remains true if we only suppose that

$$\inf_{i\in I} \frac{\chi_-^h(M_i)}{d_i} = 0,$$

and that

$$\inf_{i\in I} \frac{c_+(F_i)}{\sqrt{\ln\ln \frac{d_i}{\chi_-^h(M_i)}}} = 0,$$

where $c_+(F_i)$ is the complexity of a thin decomposition for the Heegaard splitting with minimal genus Heegaard surface F_i (see [CaGo], [ST2] and section 1). For instance, theorem 1 still holds if there exists a constant $\theta \in (0,1)$ such that $\inf_{i\in I} \frac{\chi_-^h(M_i)}{d_i^\theta} = 0$, and if $\inf_{i\in I} \frac{c_+(F_i)}{\sqrt{\ln\ln d_i}} = 0$. The first of those two last hypotheses looks quite reasonable in the light of Lackenby's gradient conjecture 1 and virtual fibration results obtained also by Lackenby in [Lac2004], especially in the case of a tower of regular finite covers.

The complexity $c_+(F_i)$ is bounded from above by the Heegaard Euler characteristic $\chi_-^h(M_i)$ of M_i, and from below by the complexity $c_+(M_i)$ of the manifold M_i itself. Nevertheless, we do not know wether a thin decomposition of minimal complexity $c_+(M_i)$ (the complexity of the cover M_i) can always be obtained from a minimal genus Heegaard splitting of M_i. We may have for all minimal genus Heegaard surface F_i of M_i, $c_+(M_i) < c_+(F_i)$. This makes the quantity $c_+(F_i)$ quite difficult to estimate. Hence it seems more natural to express the fibration criterion just in terms of the behavior of the Heegaard-Euler characteristic in finite coverings of M. This leads to the following definition.

Definition 2. *The **sub-logarithmic Heegaard gradient** of the manifold M is the quantity*

$$\nabla_{log}^h(M) = \inf\left\{\frac{\chi_-^h(M_i)}{\sqrt{\ln\ln d_i}}\right\},$$

where the infimum is taken on the (countable) set of finite coverings of M.

Similarly, one can define the ***strong sub-logarithmic Heegaard gradient*** *of M by*

$$\nabla_{log}^{sh}(M) = \inf \left\{ \frac{\chi_-^{sh}(M_i)}{\sqrt{\ln \ln d_i}} \right\},$$

where the Heegaard Euler characteristic of the finite covering $M_i \to M$ is replaced by the strong Heegaard Euler characteristic.

This definition provides an immediate corollary of theorem 1. This corollary is a sub-logarithmic version of Lackenby's conjecture 1, in the case of a closed manifold M.

Corollary 2. *The sub-logarithmic Heegaard gradient of a connected, orientable and closed hyperbolic 3-manifold M vanishes if and only if M is virtually fibred over the circle $\mathbb{S}^1$.*

The necessary condition of the corollary 2 arises from the fact that every virtually fibred 3-manifold M admits a tower of finite coverings with bounded Heegaard genus. Indeed, such a manifold M admits a finite covering $M' \to M$ where M' fibers over the circle. The Heegaard genus in the tower $(M_i \to M)_{i\in\mathbb{N}}$ of finite coverings of M' dual to the fiber F of M' is uniformly bounded above by $|\chi(F)| + 3$. Thus, for this family of coverings, we have $\lim_{i\to+\infty} \frac{\chi_-^h(M_i)}{\sqrt{\ln \ln d_i}} = 0$. This shows that the sub-logarithmic Heegaard gradient of M vanishes.

Compared to results obtained by Lackenby (see for instance [Lac2006] and [Lac2004]), the interest of theorem 1 and corollary 2 is that we do not need to consider regular covers or covers of M with bounded irregularity, as defined by Lackenby [Lac2004]. Furthermore, the family of finite covers $(M_i \to M)_{i\in I}$ of M need not be a tower of coverings nor a lattice.

With the hypothesis of theorem 1, when the index i is large enough, the manifold M_i contains a lot of virtual fibers which are incompressible embedded surfaces. We can then estimate the behavior of the strong sub-logarithmic Heegaard gradient.

Theorem 3. *Let M be a connected, orientable and closed hyperbolic 3-manifold. If there exists an infinite family of coverings $(M_i \to M)_{i\in\mathbb{N}}$ with finite covering degrees d_i such that*

$$\lim_{i\to+\infty} \frac{\chi_-^h(M_i)}{\sqrt{\ln \ln d_i}} = 0,$$

then for any $\theta \in (0, 1)$,

$$\lim_{i\to+\infty} \frac{\chi_-^{sh}(M_i)}{(\ln d_i)^\theta} = +\infty.$$

For every 3-manifold N, one has $\chi_-^h(N) \leq \chi_-^{sh}(N)$, so the sub-logarithmic Heegaard gradient is always less than or equal to the strong sub-logarithmic gradient. If the sub-logarithmic gradient is strictly positive, the strong sub-logarithmic gradient is of course also strictly positive. The following corollary can be derived from theorem 3, showing that the sub-logarithmic version of Lackenby's conjecture 2 is true.

Corollary 4. *The strong sub-logarithmic Heegaard gradient of a connected, orientable and closed hyperbolic 3-manifold M is always strictly positive.*

Plan of the paper.

In the first section, we introduce some basic definitions and properties on the setting of Heegaard splittings. In the second section, we prove theorem 1, which is the central theorem of this paper. For the proof, we need three key propositions, the proof of which we postpone to the three next sections. Finally we prove in the sixth and last section our results about strong sub-logarithmic Heegaard gradient, namely theorem 3 and corollary 4.

Acknowledgements.

I would like to thank warmly Juan Souto, Joan Porti, Anne Berry and Geneviève Simonet, Jean-Pierre Otal, Vincent Guirardel and Cyril Lecuire for very helpful conversations during the

elaboration of this paper. I add a very special thank for my advisor, Michel Boileau, whose encouragements, kindness and patience were essential ingredients in this work.

1. Background on Heegaard splittings.

In this section, we briefly summarize the theory of Heegaard splittings. We also refer to [Scha] for a survey on the subject.

A **handlebody** is the regular neighborhood of a graph. Its boundary is a connected, orientable and closed surface. The genus g of this surface is called the **genus** of the handlebody. The original graph is called a **spine** for the handlebody. If an orientable 3-manifold M is closed, a **Heegaard splitting** of M is a decomposition of M as the union of two handlebodies with the same genus, glued together by a diffeomorphism of their boundaries. A **compression body** is a connected and orientable 3-manifold H with boundary, obtained from a regular neighborhood $S \times [0,1]$ of a closed surface S, not necessarily connected. One glues some 1-handles to the surface $S \times \{1\}$ to get the compression body H. The surface $S \times \{0\}$, denoted by $\partial_- H$, is called the **negative boundary** of the compression body H. The boundary of H minus the negative boundary $\partial_- H$ is a connected surface $\partial_+ H$, called the **positive boundary** of H. The genus of the closed surface $\partial_+ H$ is called the **genus** of the compression body H and denoted by $g(H)$. By convention, if $\partial_- H$ is the empty set, the manifold H is a handlebody. A **spine** for a compression body H is the union Γ of the negative boundary $\partial_- H$ together with a graph whose vertices lie on $\partial_- H$, such that H deformation retracts on Γ. A **Heegaard splitting** of a 3-manifold M with boundary is a decomposition of M as the union of two compression bodies of the same genus glued together by a diffeomorphism of their positive boundaries. The negative boundary of one of the two compression bodies can be empty in a Heegaard splitting of M. The **genus** of a Heegaard splitting is the common genus of the compression bodies. Their common positive boundary is called a **Heegaard surface** for the manifold M.

Every compact and orientable 3-manifold M admits a Heegaard splitting. The **Heegaard genus** of the manifold M, denoted by $g(M)$, is the minimal genus of all Heegaard splittings of M. The **Heegaard Euler characteristic** of M is $\chi_-^h(M) = 2g(M) - 2$, the negative part of the Euler characteristic of a minimal genus Heegaard surface for M.

A **meridian disc** for a Heegaard splitting of M is a properly embedded disc in one of the compression bodies, which bounds an essential curve in the Heegaard surface. A Heegaard splitting (or a Heegaard surface) is said to be **strongly irreducible** if there does not exist any pair of disjoint meridian discs, one in each compression body. In other words, in a strongly irreducible Heegaard splitting, the boundaries of any two meridian discs each in one side of the Heegaard surface necessarily intersect. For any orientable 3-manifold M, one defines the **strong Heegaard Euler characteristic** $\chi_-^{sh}(M)$ of M as the minimum over all strongly irreducible Heegaard surfaces F of the negative part $\chi_-(F)$ of the Euler characteristic of F. If the manifold M does not have any strongly irreducible Heegaard splitting, then $\chi_-^{sh}(M) = +\infty$.

Note that in the case of hyperbolic 3-manifolds, the Heegaard Euler characteristics and the strong Heegaard Euler characteristics are always strictly positive.

A Heegaard splitting can be seen as a handle decomposition for a closed 3-manifold M. Starting from a collection of 0-handles, one attaches some 1-handles to them, then a collection of 2-handles, to finish by 3-handles. The first handlebody corresponds to the 0- and 1-handles, to which the 2- and 3-handles that compose the second handlebody are attached. To each 1- and 2-handles, one can associate a meridian disc. If the splitting is not strongly irreducible, two disjoint meridian discs can be used to change the order in which the handles are attached. A 2-handle corresponding to one of the meridian discs can be attached before a 1-handle corresponding to the other meridian disc. The result is called a generalised Heegaard splitting of M. More generally, a **generalised Heegaard splitting** for a 3-manifold M corresponds to a handle decomposition : starting from 0-handles and possibly collars of some boundary components of M, one attaches some 1-handles, then a collection of 2-handles, then another collection of 1-handles, and so on, alternating 1- and 2-handles, to finish after the last collection of 2-handles with a collection of 3-handles. If one stops during the process, the object obtained after attaching the j-th batch of 1- or 2-handles is a 3-manifold embedded in M. Let F_j be its boundary, after discarding any 2-sphere component that bounds a 0- or a 3-handle. After a small isotopy to make all the surfaces F_j disjoint, one gets a collection of $2n-1$ disjoint surfaces F_j in M. The surfaces F_{2j},

called the **even surfaces**, separate the manifold M into n 3-manifolds, for which the surfaces F_{2j-1}, called the **odd surfaces**, form Heegaard surfaces.

Let F be a closed and orientable surface. If F is connected, one defines the **complexity** of F as $c(F) = 0$ if F is the 2-sphere $\mathbb{S}^2$, and $c(F) = 2g(F) - 1 = 1 - \chi(F)$ otherwise. If F is not connected, the complexity of F is the sum over all components of F of the complexity of the component.

If $H = \{F_1, F_2, \ldots, F_{2n-1}\}$ is a generalised Heegaard splitting of M, the **width** of this decomposition is the set $w(H) = \{c(F_1), \ldots, c(F_{2n-1})\}$ of the complexities of the odd surfaces, with repetitions and arranged in monotonically non-increasing order.

Starting from a Heegaard splitting with Heegaard surface F which is not strongly irreducible, one can change the order in which the 1- and 2-handles are attached, to get a generalised Heegaard splitting. Each of those generalised splittings has a width. Widths can be compared using the lexicographic order. A generalised splitting obtained from F of minimal width is called a **thin position for** F (see [ST2]). The first integer of the width of a thin position for F plus one (i.e. the maximal complexity of the odd surfaces of the thin position plus one) is called the **complexity** of a thin position for F, or the complexity of the Heegaard decomposition with surface F, and denoted by $c_+(F)$. For example, if F is a strongly irreducible Heegaard surface of positive genus, then the complexity of a thin position for F is $c_+(F) = -\chi(F)$, as the original Heegaard splitting of surface F is already in a thin position. In the general case, the complexity $c_+(F)$ is always bounded from above by $-\chi(F)$ as long as the genus of F is non-negative, as each surface of a generalised Heegaard splitting arising from the splitting associated to F is obtained from F by surgery.

2. Proof of theorem 1.

In this section, we give a proof of theorem 1. This proof relies on three key propositions, the proof of which we postpone to the next three sections. We prove first theorem 1 under two weaker assumptions. At the end of the proof, we then show that those two assumptions imply the assumption stated in the theorem.

Let M be a connected, orientable and closed hyperbolic 3-manifold. Assume that there exists an infinite family $(M_i \to M)_{i \in I}$ of coverings of M with finite degrees d_i such that

$$\inf_{i \in I} \frac{\chi_-^h(M_i)}{d_i} = 0,$$

and that

$$\inf_{i \in I} \frac{c_+(F_i)}{\sqrt{\ln \ln \frac{d_i}{\chi_-^h(M_i)}}} = 0,$$

where F_i is a minimal genus Heegaard surface of M_i and $c_+(F_i)$ is the complexity of a thin position for this splitting. Without loss of generality and up to passing to a subsequence, we can assume that $I = \mathbb{N}$, that $\lim_{i \to +\infty} \frac{\chi_-^h(M_i)}{\sqrt{d_i}} = 0$ and $\lim_{i \to +\infty} \frac{c_+(F_i))}{\sqrt{\ln \ln d_i}} = 0$.

Let $i \in \mathbb{N}$, and F_i the minimal genus Heegaard surface in M_i as before. As recalled in section 1, one can use the Heegaard surface F_i to construct a generalised Heegaard splitting in a thin position. In particular, the complexity of this generalised Heegaard splitting is equal to $c_+(F_i)$. The interest of a thin position lies in the following theorem. Its topological part (1) is a consequence of the work of Casson and Gordon, Scharlemann and Thompson ([CaGo] and [ST2]). The metric part (2) arises from results of Frohman, Freedman, Hass and Scott for incompressible surfaces ([FHS] and [FrHa]). The last part (3) is a result of Pitts and Rubinstein ([PiRu], see also [Sou], [CoDL] and [Pi]).

Theorem 5. *Let N be a connected, orientable and compact hyperbolic 3-manifold, and H a generalised Heegaard splitting that is a thin position for M. Then this generalised splitting has the following three properties.*

1. *Every even surface is incompressible in N and the odd surfaces form strongly irreducible Heegaard surfaces for the components of N cut along the even surfaces.*
2. *All the even surfaces are isotopic to minimal surfaces or to the boundary of a regular neighborhood of a non-orientable minimal surface.*

(3) *Every odd surface is isotopic to a minimal surface or to the boundary of a regular neighborhood of a non-orientable minimal surface with a small tube attached vertically in the I-bundle structure.*

In the sequel, for brevity we will call a **pseudo-minimal surface** a surface which is the boundary of a regular neighborhood of a minimal non-orientable surface, possibly with a little tube attached vertically.

In our case, we have the following result.

Lemma 6. *For all $i \in \mathbb{N}$, there exists a thin position for M_i obtained from the minimal genus Heegaard surface F_i such that every even surface is a minimal surface or the boundary of a small regular neighborhood of a non-orientable minimal surface, and every odd surface is a minimal surface or the boundary of a regular neighborhood of a non-orientable minimal surface with a small tube attached vertically in the I-bundle structure.*

Moreover, for all $i \in \mathbb{N}$, there exists a compression body C_i among the compression bodies of this thin position such that

$$\mathrm{Vol}(C_i) \geq \mathrm{Vol}(M)\,\frac{d_i}{\chi_-^h(M_i)+2}.$$

Proof of lemma 6.

First, we use part (2) of theorem 5 to isotope the even surfaces of a thin position obtained from the Heegaard surface F_i to minimal surfaces or to the boundary of a regular neighborhood of a non-orientable minimal surface.

By part (3) of theorem 5, after isotopy we can assume that the odd surfaces are minimal surfaces, or the boundary of a regular neighborhood of a non-orientable minimal surface with a small tube attached vertically in the I-bundle structure.

The number $\mathcal{N}_i$ of compression bodies arising from a generalised Heegaard splitting of M_i obtained from the Heegaard surface F_i is less than or equal to the total number of 1- and 2-handles of a handle decomposition associated to the Heegaard surface F_i : $\mathcal{N}_i \leq 2g(F_i) = |\chi(F_i)| + 2 = \chi_-^h(M_i) + 2$.

There exists at least one compression body C_i such that

$$\mathrm{Vol}(C_i) \geq \frac{\mathrm{Vol}(M_i)}{\mathcal{N}_i} \geq \mathrm{Vol}(M)\,\frac{d_i}{\chi_-^h(M_i)+2}.$$

For each i, chose such a compression body C_i to obtain lemma 6. $\square$

For all $i \in \mathbb{N}$, take a compression body C_i as in lemma 6.

Definition 3. *Let x be a point in C_i and S an immersed surface in C_i. We say that S **separates** x **from** $\partial_+ C_i$ if every oriented path from x to $\partial_+ C_i$ has its algebraic intersection number equal to $+1$.*

*If two surfaces S and T immersed in C_i are such that S separates every point of T from $\partial_+ C_i$, we say that T **separates** S **from** $\partial_+ C_i$. In this case, the surfaces S and T are said to be **nested**.*

We recall that the genus of the compression body is $g(C_i) = g(\partial_+ C_i)$, the genus of its positive boundary.

We will denote the ceil function of the real number x by $\lceil x \rceil$, i.e. the smallest integer not less than x. Similarly, $\lfloor x \rfloor$ is the floor function of x, and represents the largest integer no greater than x. By convention, we set $\lceil x \rceil$ and $\lfloor x \rfloor$ equal to zero if x is non-positive.

The following proposition is a step towards the construction of a certain amount of parallel surfaces in the compression body C_i. It is an adaptation of Lemma 4.5 p. 2251 of [Mah]. We postpone its proof to section 3.

Proposition 7 (Nested Surfaces)**.** *Let δ_i be the diameter of the compression body C_i of M_i. Let ϵ be the injectivity radius of M, $K_i = 4\left(3 + \frac{1}{\sinh \frac{\epsilon}{8}^2}\right) g(C_i) - 10$ and $K_i' = 2a' \,|\chi(\partial_+ C_i)|$, with $a' = 6(\frac{21}{4} + \frac{3}{4\pi} + \frac{3}{4\epsilon} + \frac{2}{\sinh^2(\epsilon/4)})$.*

For every $i \in \mathbb{N}$, there exist at least $n_i = \lceil \frac{\delta_i}{36\epsilon K_i} - \frac{2}{9} - \frac{K_i'}{9K_i} \rceil$ disjoint nested surfaces immersed in C_i. All of those surfaces are homotopic to surfaces obtained from compressions of $\partial_+ C_i$.

Moreover, the diameter of those surfaces is bounded from above by $2\epsilon K_i$ and they are separated from each other by a distance greater than or equal to $10\epsilon K_i$.

With this proposition, for each $i \in \mathbb{N}$ we obtain at least $n_i = \lceil \frac{\delta_i}{36\epsilon K_i} - \frac{2}{9} - \frac{K'_i}{9K_i} \rceil$ nested immersed surfaces in the handlebody C_i. Those surfaces are all disjoint and obtained from $\partial_+ C_i$ by surgery. This implies that the the genus of those surfaces is between 0 and $g(C_i)$, the genus of C_i (which is, by definition of a thin position, less than or equal to $g(F_i) = g(M_i)$). We can thus find at least $\lfloor \frac{n_i}{g(C_i)+1} \rfloor$ such nested immersed surfaces of the same genus, where $\lfloor x \rfloor$ is the floor function of the real number x, with the convention that $\lfloor x \rfloor$ is zero if x is non-positive. The next step is then to replace those nested immersed surfaces by parallel embedded surfaces.

Proposition 8 (Parallel Surfaces). *Let δ_i be the diameter of the compression body C_i in M_i. Let ϵ be the injectivity radius of M, $K_i = 4\left(3 + \frac{1}{\sinh \frac{\epsilon}{8}^2}\right) g(C_i) - 10$ and $K'_i = 2a' \left|\chi(\partial_+ C_i)\right|$, with $a' = 6(\frac{21}{4} + \frac{3}{4\pi} + \frac{3}{4\epsilon} + \frac{2}{\sinh^2(\epsilon/4)})$.*

For every $i \in \mathbb{N}$, there exist at least $m_i = (\lfloor \frac{1}{g(C_i)+1} \lceil \frac{\delta_i}{36\epsilon K_i} - \frac{2}{9} - \frac{K'_i}{9K_i} \rceil \rfloor - 4)$ parallel and connected surfaces embedded in C_i, whose diameter in the manifold M_i is uniformly bounded from above by $9\epsilon K_i$ and separated from each other by a distance greater than or equal to ϵK_i.

For the proof of this proposition, see section 4. Those m_i parallel surfaces are candidates for a virtual fiber. But we still have to select some of them to get a virtual fibration of the base manifold M.

Let $\mathcal{D}$ be a Dirichlet fundamental polyhedron for M in its universal cover $\widehat{M} \simeq \mathbb{H}^3$. Translates of $\mathcal{D}$ by the covering transformations give a tiling of the universal cover $\widehat{M}$. This tiling descends to a tiling of the finite cover M_i by d_i copies of $\mathcal{D}$. Each of the m_i parallel, connected and embedded surfaces in M_i obtained by proposition 8 intersects a finite and connected set of copies of $\mathcal{D}$. We call such a set a **pattern of fundamental domains**. We can suppose that each of the embedded surfaces is transverse to the 2-skeleton of the tiling. More precisely, we can suppose that each surface does not meet the vertices of the fundamental polyhedra, that it intersects the edges in isolated points and it is transverse to the 2-dimensional faces of the polyhedra. Thus a pattern of fundamental domains is a connected set that is the union of copies of $\mathcal{D}$ glued along some of their 2-dimensional faces.

Lemma 9. *Let $\mathcal{D}$ be a Dirichlet fundamental polyhedron for M in $\mathbb{H}^3$. Let α be the number of faces of $\mathcal{D}$ of dimension two, and ι the maximum over all pairs (F_1, F_2) of the number of orientation-reversing isometries between the 2-dimensional faces F_1 and F_2 of the polyhedron $\mathcal{D}$ (F_1 and F_2 are not necessarily distinct).*

For each $\ell \in \mathbb{N}$, the number of possibilities to glue together ℓ copies of $\mathcal{D}$ to form a pattern of ℓ fundamental domains is less than or equal to $\left(\alpha! \sqrt{(\alpha+1)\iota^{\alpha}}\right)^{(\ell+1)^2}$.

Proof of lemma 9.

For every $\ell \in \mathbb{N}$, let us denote by $B(\ell)$ the number of possibilities to glue together ℓ copies of $\mathcal{D}$ to form a pattern of ℓ fundamental domains. We have to find an upper bound for the number of possibilities to identify pairwise some 2-dimensional faces of ℓ Dirichlet polyhedra.

First, let $(\mathcal{D}_1, \mathcal{D}_2)$ be a pair of copies of the fundamental domain $\mathcal{D}$. We want to estimate the number of possibilities to glue those two polyhedra together. As each polyhedron has α 2-dimensional faces, a gluing of this pair is given by k triples $\{(F^1_j, F^2_j, h_j)\}_{j=1\ldots k}$, where k is an integer between 0 and α, and for all $j \in \{1, \ldots, k\}$, the face F^1_j of $\mathcal{D}_1$ is identified with the face F^2_j of $\mathcal{D}_2$ by the orientation-reversing isometry $h_j : F^1_j \longrightarrow F^2_j$.

Remember that ι is the maximum over all pairs (F_1, F_2) of the number of orientation-reversing isometries between the two (non necessarily distinct) 2-dimensional faces F_1 and F_2 of the polyhedron $\mathcal{D}$. If $n(\mathcal{D}_1, \mathcal{D}_2)$ is the number of possibilities for gluing together the two polyhedra $\mathcal{D}_1$ and $\mathcal{D}_2$, we have the following upper bound :

$$\begin{aligned} n(\mathcal{D}_1, \mathcal{D}_2) &\leq 1 + \alpha^2 \iota + (\alpha(\alpha-1))^2 \iota^2 + \ldots + (\alpha!)^2 \iota^\alpha \\ &\leq \sum_{k=0}^{\alpha} \left(\frac{\alpha!}{(\alpha-k)!} \right)^2 \iota^k \\ &\leq (\alpha+1)(\alpha!)^2 \iota^\alpha. \end{aligned}$$

By definition, the function $B(\ell)$ is the number of possibilities for gluing together ℓ polyhedra $\mathcal{D}_1, \ldots, \mathcal{D}_\ell$ all isomorphic to $\mathcal{D}$. From the former estimation, we get :

$$\begin{aligned} B(\ell) &\leq \prod_{1 \leq j \leq k \leq \ell} n(\mathcal{D}_j, \mathcal{D}_k) \\ &\leq \prod_{1 \leq j \leq k \leq \ell} (\alpha+1)(\alpha!)^2 \iota^\alpha \\ &\leq \left((\alpha+1)(\alpha!)^2 \iota^\alpha \right)^{\ell(\ell+1)/2} \\ &\leq \left((\alpha+1)(\alpha!)^2 \iota^\alpha \right)^{(\ell+1)^2/2}, \end{aligned}$$

so we finally have :

$$B(\ell) \leq \left(\alpha! \sqrt{(\alpha+1)\iota^\alpha} \right)^{(\ell+1)^2}.$$

□

The following key proposition is a quantitative version of Lemma 4.12 p. 2258 of [Mah]. We postpone its proof to section 5.

Proposition 10 ("Pattern Lemma"). *Assume that in the cover M_i we have m_i connected, separating, orientable, embedded and disjoint parallel surfaces, with diameter at most λ and at distance at least $r > 0$ from each other.*

Let $\mathcal{D}$ be a Dirichlet fundamental domain for the manifold M in its universal cover $\widehat{M} \simeq \mathbb{H}^3$. Let us denote by D the diameter of $\mathcal{D}$, α the number of its 2-dimensional faces and β the number of faces of dimension zero, one and two.

For all $\ell \in \mathbb{N}$, let $B(\ell)$ be an upper bound for the number of possibilities of patterns obtained by gluing together at most ℓ fundamental domains that intersect a connected, orientable and embedded surface. Let $L = \lceil \frac{\beta\lambda}{\sigma} \rceil$, where σ is the minimum over all pairs of non-adjacent faces of $\mathcal{D}$ of the distance between those faces. (As $\mathcal{D}$ is compact, σ is a well defined strictly positive real number.)

If $r/2D > 1$ and $\frac{m_i}{\alpha^2 L^2 B(L)} \geq 7$, or if $r/2D < 1$ and $\left(\frac{r}{2D+1} m_i - 1 \right) \frac{1}{\alpha^2 L^2 B(L)} \geq 7$, then the manifold M virtually fibers over the circle $\mathbb{S}^1$, and the m_i parallel surfaces are virtual fibers.

Remark 1. *The square root and the first logarithm of the denominator in the sub-logarithmic Heegaard gradient arise from the use of lemma 9 in the proof of this proposition.*

We can now finish the proof of theorem 1. Suppose that we have an infinite family of finite coverings $\{M_i \to M\}_{i \in \mathbb{N}}$ such that

$$\lim_{i \to +\infty} \frac{\chi_-^h(M_i)}{d_i} = 0, \tag{1}$$

and that

$$\lim_{i \to +\infty} \frac{c_+(F_i)}{\sqrt{\ln \ln \frac{d_i}{\chi_-^h(M_i)}}} = 0. \tag{2}$$

We can apply proposition 10 with the number m_i of parallel and embedded surfaces given by proposition 8 :

$$m_i = \lfloor \frac{1}{g(C_i)+1} \lceil \frac{\delta_i}{36\epsilon K_i} - \frac{2}{9} - \frac{K_i'}{9K_i} \rceil \rfloor - 4,$$

$$r = \epsilon K_i,$$
$$\lambda = 9\epsilon K_i,$$

where

$$K_i = 4\left(3 + \frac{1}{\sinh \frac{\epsilon}{8}^2}\right) g(C_i) - 10,$$

and

$$K_i' = 2a' \left|\chi(\partial_+ C_i)\right|,$$

with

$$a' = 6(\frac{21}{4} + \frac{3}{4\pi} + \frac{3}{4\epsilon} + \frac{2}{\sinh^2(\frac{\epsilon}{4})}).$$

We have then $L_i = \lceil \frac{9\beta\epsilon K_i}{\sigma} \rceil$. In particular, $9\beta\epsilon K_i/\sigma + 1 \geq L_i \geq 9\beta\epsilon K_i/\sigma$.

Let us show that the diameter δ_i of the compression body C_i tends to infinity when i tends to infinity. On the one hand, we have

$$\mathrm{Vol}(C_i) \leq \mathrm{Vol}\left(\mathbb{B}^{\mathbb{H}^3}\left(\frac{\delta_i}{2}\right)\right) = \pi(\sinh(\delta_i) - \delta_i) \leq \frac{\pi}{2} e^{\delta_i}.$$

Remark 2. *The second logarithm of the denominator in the sub-logarithmic Heegaard gradient comes from this estimation linking the diameter with the volume of a hyperbolic 3-manifold.*

On the other hand, lemma 6 give the lower bound

$$\mathrm{Vol}(C_i) \geq \frac{\mathrm{Vol}(M_i)}{\chi_-^h(M_i) + 2} = \mathrm{Vol}(M)\,\frac{d_i}{\chi_-^h(M_i) + 2},$$

and by assumption (1), the right hand side tends to infinity when the integer i tends to infinity, showing that δ_i tends also to infinity. Furthermore, we obtain then the following majoration :

$$\delta_i \geq \ln\left(\frac{d_i}{\chi_-^h(M_i) + 2}\right) + \ln\left(\frac{2\mathrm{Vol}(M)}{\pi}\right). \tag{3}$$

If the genus $g(C_i)$ of the compression body C_i is uniformly bounded with respect to i, the quantity K_i is bounded from above, and m_i tends to infinity when i tends to infinity. The quotient

$$\min\left(m_i, \frac{r}{2D+1} m_i - 1\right) \frac{1}{\alpha^2 L_i^2 B(L_i)}$$

tends then to infinity. For i large enough, this quotient is greater than 7, and by proposition 10, the manifold M_i contains then a virtual fiber that is an embedded surface.

Otherwise, up to passing to a subsequence, we can suppose that $\lim_{i\to+\infty} g(C_i) = +\infty$, thus that $K_i \to +\infty$. For i large enough, we have $r = \epsilon K_i > 2D$. To be in the first case of proposition 10, it suffices to prove that the quantity $\frac{m_i}{\alpha^2 L_i^2 B(L_i)}$ tends to infinity when i tends to infinity.

Let $G(\ell) = \alpha^2 \ell^2 (\alpha! \sqrt{(\alpha+1)\iota^\alpha})^{(\ell+1)^2}$: from lemma 9, we get $\alpha^2\ell^2 B(\ell) \leq G(\ell)$ for all positive integer ℓ, so it is sufficient to show that $\frac{m_i}{G(L_i)}$ tends to infinity when i tends to infinity.

Let

$$R_i = \frac{m_i}{G(L_i)} = \left(\lfloor \frac{1}{g(C_i)+1} \lceil \frac{\delta_i}{36\epsilon K_i} - \frac{2}{9} - \frac{K_i'}{9K_i} \rceil \rfloor - 4\right) \frac{1}{\alpha^2 L_i^2 (\alpha! \sqrt{(\alpha+1)\iota^\alpha})^{(L_i+1)^2}}.$$

$$R_i \geq \left(\frac{1}{2(g(C_i)+1)}\left(\frac{\delta_i}{18\epsilon K_i} - \frac{4}{9} - \frac{2K_i'}{9K_i}\right) - 5\right) \frac{1}{\alpha^2 (9\beta\epsilon K_i/\sigma + 1)^2 (\alpha! \sqrt{(\alpha+1)\iota^\alpha})^{(9\beta\epsilon K_i/\sigma+2)^2}}.$$

As $2 \leq g(C_i) \leq \frac{c_+(F_i)}{2} + 1$, if we denote by $a = 2(3 + \frac{1}{\sinh(\frac{\epsilon}{8})^2})$ and $b = 2(1 + \frac{2}{\sinh(\frac{\epsilon}{8})^2})$, we have $2a + b \leq K_i \leq ac_+(F_i) + b$.

Since $\lim_{i\to+\infty} g(C_i) = +\infty$,

$$\frac{2K_i'}{9K_i} = \frac{2 \times 2a'(2g(C_i) - 2)}{9(a(2g(C_i) - 2) + b)} \sim_{i\to+\infty} \frac{4a'}{9a}.$$

Taking also the inequality (3) into account, for i large enough such that $\frac{2K_i'}{9K_i} \leq \frac{5a'}{9a}$, we get :

$$R_i \geq \left(\frac{1}{c_+(F_i)+4} \left(\frac{\ln\left(\frac{d_i}{\chi_-^h(M_i)+2}\right) + \ln\left(\frac{2\mathrm{Vol}(M)}{\pi}\right)}{18\epsilon a c_+(F_i) + 18\epsilon b} - \frac{4a+5a'}{9a} \right) - 5 \right) \frac{1}{G_i},$$

where

$$G_i = \alpha^2(9\beta\epsilon a/\sigma c_+(F_i) + 9\beta\epsilon b/\sigma + 1)^2(\alpha!\sqrt{(\alpha+1)\iota^\alpha})^{(9\beta\epsilon a/\sigma c_+(F_i)+9\beta\epsilon b/\sigma+2)^2}.$$

Let $R'_i = \left(\frac{1}{c_+(F_i)+4} \left(\frac{\ln\left(\frac{d_i}{\chi_-^h(M_i)+2}\right) + \ln\left(\frac{2\mathrm{Vol}(M)}{\pi}\right)}{18\epsilon a c_+(F_i) + 18\epsilon b} - \frac{4a+5a'}{9a} \right) - 5 \right) \frac{1}{G_i}$. As $R_i \geq R'_i$, it suffices to prove that R'_i tends to infinity when i tends to infinity. Let us look for an equivalent of R'_i when i tends to infinity.

For every i, $2g(C_i) - 2 \leq c_+(F_i) \leq \chi_-^h(M_i)$. As we have supposed that $\lim_{i\to+\infty} g(C_i) = +\infty$, $\lim_{i\to+\infty} c_+(F_i) = +\infty$ and $\lim_{i\to+\infty} \chi_-^h(M_i) = +\infty$. Moreover,

$$\frac{\ln\left(\frac{d_i}{\chi_-^h(M_i)+2}\right) + \ln\left(\frac{2\mathrm{Vol}(M)}{\pi}\right)}{18\epsilon a c_+(F_i) + 18\epsilon b} \sim_{i\to+\infty} \frac{\ln\left(\frac{d_i}{\chi_-^h(M_i)}\right)}{18\epsilon a c_+(F_i)}.$$

From assumption (2), we have

$$\lim_{i\to\infty} \frac{c_+(F_i)}{\sqrt{\ln\ln \frac{d_i}{\chi_-^h(M_i)}}} = 0.$$

Hence

$$\lim_{i\to\infty} \frac{\ln\left(\frac{d_i}{\chi_-^h(M_i)+2}\right) + \ln\left(\frac{2\mathrm{Vol}(M)}{\pi}\right)}{18\epsilon a c_+(F_i) + 18\epsilon b} = \lim_{i\to\infty} \frac{\ln\left(\frac{d_i}{\chi_-^h(M_i)}\right)}{18\epsilon a c_+(F_i)} = +\infty.$$

Thus,

$$R'_i \sim \left(\frac{1}{c_+(F_i)} \times \frac{\ln\left(\frac{d_i}{\chi_-^h(M_i)}\right)}{18\epsilon a c_+(F_i)} - 5 \right) \frac{1}{G_i}.$$

Always from assumption (2),

$$\lim_{i\to+\infty} \frac{\ln\left(\frac{d_i}{\chi_-^h(M_i)}\right)}{c_+(F_i)^2} = +\infty$$

and so

$$R'_i \sim \frac{\ln\left(\frac{d_i}{\chi_-^h(M_i)}\right)}{18\epsilon a c_+(F_i)^2 G_i}$$

when i tends to infinity.

Therefore, it suffices to prove that $R''_i = \frac{\ln\left(\frac{d_i}{\chi_-^h(M_i)}\right)}{c_+(F_i)^2 G_i}$ tends to infinity when i tends to infinity. But

$$\begin{aligned}
\ln R''_i &= \ln\ln\left(\frac{d_i}{\chi_-^h(M_i)}\right) - 2\ln c_+(F_i) - \ln G_i \\
&= \ln\ln\left(\frac{d_i}{\chi_-^h(M_i)}\right) - 2\ln c_+(F_i) - 2\ln\alpha - 2\ln(9\beta\epsilon a/\sigma c_+(F_i) + 9\beta\epsilon b/\sigma + 1) \\
&\quad -(9\beta\epsilon a/\sigma c_+(F_i) + 9\beta\epsilon b/\sigma + 2)^2 \ln(\alpha!\sqrt{(\alpha+1)\iota^\alpha}).
\end{aligned}$$

As $c_+(F_i) \to +\infty$ when i tends to infinity, we have :

$$\ln R''_i = \ln\ln\left(\frac{d_i}{\chi_-^h(M_i)}\right) - \left((9\beta\epsilon a/\sigma)^2 \ln(\alpha!\sqrt{(\alpha+1)\iota^\alpha})\right) c_+(F_i)^2 + o\left(c_+(F_i)^2\right).$$

As after assumption (2),

$$\lim_{i\to+\infty} \frac{c_+(F_i)^2}{\ln\ln\left(\frac{d_i}{\chi_-^h(M_i)}\right)} = 0,$$

we have $\lim_{i\to+\infty} R_i'' = +\infty$, which ends the proof of theorem 1 under assumptions (1) and (2).

To prove theorem 1, it remains to show that the assumption of this theorem implies assumptions (1) and (2).

Lemma 11. *Let M be a connected, orientable and closed hyperbolic 3-manifold. Suppose that there exists a family $\{M_i \to M\}_{i\in\mathbb{N}}$ of coverings of M with finite degrees d_i such that $\lim_{i\to+\infty} \frac{\chi_-^h(M_i)}{\sqrt{\ln\ln d_i}} = 0$. Then assumptions (1) and (2) are satisfied : $\lim_{i\to+\infty} \frac{\chi_-^h(M_i)}{d_i} = 0$ and $\lim_{i\to+\infty} \frac{c_+(F_i)}{\sqrt{\ln\ln d_i}} = 0$, where $c_+(F_i)$ is the complexity of a thin position for a minimal genus Heegaard surface F_i of M_i.*

Proof of lemma 11.
Let us assume that

$$\lim_{i\to+\infty} \frac{\chi_-^h(M_i)^2}{\ln\ln d_i} = 0.$$

In particular, we have $\frac{\chi_-^h(M_i)}{d_i} \to 0$ and $\frac{\chi_-^h(M_i)}{\sqrt{d_i}} \to 0$ when i tends to infinity. For every i large enough, the inequality $\chi_-^h(M_i) \le \sqrt{d_i}$ is satisfied. As we always have $c_+(F_i) \le \chi_-^h(M_i)$, we obtain then :

$$\begin{aligned}
\frac{c_+(F_i)^2}{\ln\ln\left(\frac{d_i}{\chi_-^h(M_i)}\right)} &\le \frac{\chi_-^h(M_i)^2}{\ln\ln\left(\frac{d_i}{\chi_-^h(M_i)}\right)} \\
&\le \frac{\chi_-^h(M_i)^2}{\ln\ln\left(\sqrt{d_i}\right)} = \frac{\chi_-^h(M_i)^2}{\ln\left(\frac{1}{2}\ln d_i\right)} \\
&\le \frac{\chi_-^h(M_i)^2}{\ln\ln d_i - \ln 2} \longrightarrow 0
\end{aligned}$$

when i tends to infinity. □

Therefore, the hypothesis of theorem 1 is stronger and implies assumptions (1) and (2). This ends the proof of theorem 1. □

Remark 3. *If there exists a constant $\theta \in (0,1)$ such that $\lim_{i\to+\infty} \frac{\chi_-^h(M_i)}{d_i^\theta} = 0$ and if $\lim_{i\to+\infty} \frac{c_+(F_i)}{\sqrt{\ln\ln d_i}} = 0$, as before, for i large enough, $\chi_-^h(M_i) \le d_i^\theta$. We have then*

$$\begin{aligned}
\frac{c_+(F_i)^2}{\ln\ln\left(\frac{d_i}{\chi_-^h(M_i)}\right)} &\le \frac{c_+(F_i)^2}{\ln\ln\left(d_i^{1-\theta}\right)} \\
&\le \frac{c_+(F_i)^2}{\ln\ln d_i + \ln(1-\theta)} \longrightarrow 0
\end{aligned}$$

when i tends to infinity, which shows that assumptions (1) and (2) are satisfied. Thus, the conclusions of theorem 1 are still true under those assumptions.

3. Finding nested surfaces.

This section is devoted to the proof of proposition 7.

If N is an orientable and compact 3-manifold, a Heegaard splitting for N induces a **sweepout** for the manifold N, i.e. a one-parameter family $\{F_t\}_{t\in[0,1]}$ such that F_0 is a spine for the first compression body, F_1 a spine for the second compression body, each F_t for $t \in (0,1)$ is a Heegaard surface for N homeomorphic to a closed and oriented surface F, and the map $F \times [0,1] \to N$ has homological degree one.

Let $(C_i)_{i\in\mathbb{N}}$ be the sequence of compression bodies in M_i obtained in lemma 6. This means that the negative boundary of C_i is a union (possibly empty) of minimal incompressible surfaces or of boundaries of small neighborhoods of non-orientable minimal surfaces, and the positive boundary ∂_+C_i is a minimal surface or the boundary of a small regular neighborhood of a

non-orientable minimal surface with a small tube attached vertically in the I-bundle structure. Moreover, lemma 6 ensures that $\mathrm{Vol}(C_i) \geq \mathrm{Vol}(M)\frac{d_i}{\chi_-^h(M_i)+2}$ for all $i \in \mathbb{N}$.

The generalised Heegaard splitting induces a sweepout of the compression body C_i. More precisely, there exists a one-parameter family of surfaces $\{S_t\}_{t\in[0,1]}$ such that S_0 is a spine for C_i, $S_1 = \partial_+ C_i$, for all $t \in (0,1]$ the surface S_t is homeomorphic to $S = \partial_+ C_i$, and the map $\Phi : S \times I \to C_i$ is of homological degree one. This one-parameter family is called a **sweepout** for C_i.

The sweepout surfaces S_t for $t > 0$ are of interest in order to construct a long product in the compression body C_i. But, if we can control the diameter of a minimal surface in terms of its genus and the injectivity radius of the ambiant manifold, we cannot control uniformly the diameter of all the sweepout surfaces S_t : there may appear some long and thin Margulis tubes, containing a closed geodesic of the surface with length less than the injectivity radius of M_i.

Rather than the diameter, it is more convenient to work with a notion of diameter for which non-connected surfaces with small diameter components are considered as small : this notion is called the ϵ-diameter.

Definition 4. *Let $\epsilon > 0$. The **ϵ-diameter** of a non-necessarily connected surface F is the minimal number of balls of radius ϵ for the metric of F required to cover the surface F.*

In the sequel, we set ϵ to be the injectivity radius of M. As for each $i \in \mathbb{N}$, the manifold M_i is a covering of M, the constant ϵ provides a uniform lower bound for the injectivity radius of M_i.

At this point, we recall the technique of Maher to construct from the original sweepout $\{S_t\}_{t\in I}$ of C_i what he calls a "generalized sweepout" $\{\widehat{S}_t\}_{t\in I}$ in which he controls the ϵ-diameter of each sweepout surface (see [Mah, Sections 2 and 3]).

The first step is to straighten the sweepout $\{S_t\}_{t\in I}$ to a simplicial sweepout, using results of Bachman, Cooper and White [BCW]. We recall terminology and results stated in [Mah, Sections 2 and 3].

Definition 5. *A **coned** n**-simplex** in a compact Riemannian manifold N of sectional curvature at most -1 is defined inductively as follows. A **coned 1-simplex** $\Delta^1 = (v_0, v_1)$ is a constant speed geodesic from v_0 to v_1. The speed is allowed to be zero, and in this case the 1-simplex degenerates to the point v_0. A **coned** n**-simplex** is a map $\phi : \Delta^n \to N$ such that $\phi_{|\Delta^{n-1}}$ is a coned $(n-1)$-simplex and for all $x \in \Delta^{n-1}$, $\phi_{|\{tx+(1-t)v_n \,|\, t\in[0,1]\}}$ is a constant speed geodesic. The map ϕ depends on the order of the vertices $(v_0, \dots, v_n)$ and its image may not be embedded in N, just immersed.*

*A **simplicial surface** is a continuous map $\phi : S \to N$ where S is a triangulated surface, such that the restriction of the map ϕ to each triangle Δ of S is a coned 2-simplex.*

*A **simplicial sweepout** is a sweepout $\Phi : S \times I \to N$ such that each surface S_t is mapped to a simplicial surface with at most $4g(S)$ triangles, and at most one vertex of angle sum less than 2π.*

First, we need to work in a complete Riemannian manifold of sectional curvature at most -1. To this aim, we start with the compression body C_i and its non complete induced hyperbolic metric. If necessary, we need to modify slightly the compression body C_i in order that each boundary component of C_i has its intrinsic sectional curvature at most -1.

That is not a problem for boundary components which are minimal surfaces, as their sectional curvature is always at most -1.

If a boundary component of C_i is the boundary of a small neighborhood of a non-orientable minimal surface, we can choose this neighborhood small enough in order that the sectional curvature of this pseudo minimal surface is bounded from above by $-1/2$. This is a consequence of the continuity of the intrinsic sectional curvature in a neighborhood of the minimal surface (because of the continuity of the Gauss curvature). By rescaling all the metrics of the coverings M_i and the metric of M by a factor 2, we can suppose that the intrinsic curvature of the boundary components of C_i in M_i is at most -1.

If $\partial_+ C_i$ is the boundary of a regular neighborhood $N(S)$ of a non orientable minimal surface S with a small tube attached vertically in the I-bundle structure, we have to consider two cases. If this tube $\mathbb{D}^2 \times I$ belongs to the compression body C_i, we can remove it. More precisely, we compress C_i along the disc $\mathbb{D}^2 \times \{1/2\}$ to get a new compression body of lower genus. We lose the tube $\mathbb{D}^2 \times I$ during this process, but as we can make this tube as small as we like, this

compression does not change significantly the volume of the compression body. As the positive boundary of this new compression body C_i' is then the boundary of a small regular neighborhood of the minimal non orientable surface S, the previous argument shows that we can suppose that the intrinsic curvature of $\partial_+ C_i'$ is at most -1.

Otherwise, the tube $\mathbb{D}^2 \times I$ lies outside C_i, meaning that C_i is contained in $N(S)$. We can then collapse the small tube to an arbitrarily small geodesic arc γ in the regular neighborhood of the minimal non orientable surface S. The positive boundary $\partial_+ C_i$ becomes the union of the boundary of $N(S)$ and the arc γ. As before, we can suppose that the sectional curvature of $\partial N(S)$ is at most -1.

For each boundary component F of C_i, we glue a copy of $F \times [0, +\infty)$ equipped with a warped product metric. A computation of the sectional curvature of a warped product (see for example Bishop and O'Neil [BO'N, p. 26]) shows that as we start from a surface F with sectional curvature at most -1, there exists a warped product metric on $S \times [0, +\infty)$ such that this Riemannian manifold is complete of sectional curvature at most -1. If we are in the last case where F is the boundary of a regular neighborhood $N(S)$ of a minimal non orientable surface S with a small tube attached, and this tube is lying outside C_i, then we forget the arc γ for this construction and we just glue a copy of $\partial N(S) \times [0, +\infty)$ with a Riemannian metric of curvature at most -1. We perturb this metric to make it smooth, and we obtain thus a complete Riemannian metric for the interior of C_i (union γ if we are in this last case) with sectional curvature at most -1.

The boundary surfaces of C_i are minimal surfaces or pseudo minimal surfaces. This fact is crucial as one can homotop a minimal surface of genus g to a simplicial surface not too far away in C_i. This can be done by the following lemmas.

Lemma 12. *(Maher* [Mah, Lemma 4.2 p. 2249] *and* [Lac2006, Proposition 6.1]*.)*
Suppose S is a minimal surface in a closed Riemannian manifold N of sectional curvature at most -1. Let ϵ be a lower bound for the injectivity radius of N and

$$a' = 2\left(\frac{21}{4} + \frac{3}{4\pi} + \frac{3}{4\epsilon} + \frac{2}{\sinh^2(\frac{\epsilon}{4})}\right).$$

Then the surface S has ϵ-diameter at most $a'\,|\chi(S)|$, and it admits a one-vertex triangulation in which each edge has length at most $2\epsilon a'\,|\chi(S)|$.

Proof of lemma 12.

This lemma is a direct consequence of [Mah, Lemma 4.2 p. 2249] and [Lac2006, Proposition 6.1] in the case the minimal surface S is orientable, and we can take $a'/2$ instead of a'. If S is not orientable, its homology class $[S]$ is non zero in $H_2(N, \mathbb{Z}/2\mathbb{Z})$. By Poincaré's duality, it corresponds to a non-trivial element $\alpha \in H^1(N, \mathbb{Z}/2\mathbb{Z})$. As the homology class of the double cover of S can be represented by the boundary of a small regular neighborhood of the non-orientable surface S, we have $2[S] = 0$ in $H_2(N, \mathbb{Z})$. If we take the double cover N' of N corresponding to the kernel of α, the surface S lifts to a minimal orientable surface S'. We can apply the stronger version of lemma 12, and bound the ϵ-diameter of S' by $a'/2\,|\chi(S')| = a'/2 \times 2\,|\chi(S)| = a'\,|\chi(S)|$, and the length of a one-vertex triangulation for S' by $2\epsilon a'\,|\chi(S)|$. As those numbers bound also from above the ϵ-diameter and the length of a one-vertex triangulation of S, this finishes the proof of lemma 12. □

Of course, if we replace a' by $3a'$ in lemma 12, we obtain again an upper bound for the ϵ-diameter and the length of a one-vertex triangulation of the pseudo minimal surfaces which are components of the boundary of C_i, as in lemma 12 (and we can make the geodesic arcs as small as necessary).

Lemma 13. *Let N be a complete Riemannian manifold with sectional curvature at most -1. Suppose S is a connected and orientable minimal or pseudo minimal surface in N with diameter bounded from above by $\mathcal{N}$ and admitting a one-vertex triangulation in which the length of the edges is at most $\mathcal{N}'$. Then S can be homotoped to a simplicial surface S' with diameter at most $2\mathcal{N}'$ and such that any point $x \in S$ and $x' \in S'$ are at distance at most $\mathcal{N} + \mathcal{N}'$ from each other.*

Proof of lemma 13.

Let v be the vertex of the one-vertex triangulation of S. First, we homotop each edge e of the triangulation of S to its closed length-minimizing geodesic representative e' in $\pi_1(N,v)$. If the homotopy class of e is zero (meaning that the surface S is compressible in N), we homotop e to the degenerate constant speed geodesic $\{v\}$.

Let T be a triangle in S. To each edge of T corresponds a coned 1-simplex. We choose one of those 1-simplices and we cone the opposite vertex to this edge. More precisely, if at least one edge of T corresponds to a nulhomotopic curve, then we build a simplicial triangle T' corresponding to T which degenerates to a point if all edges of T are nulhomotopic, or to the corresponding closed geodesic if at least one of the edges of T is not nulhomotopic. In this case, each point in T' is at distance at most $\mathcal{N}'/2$ from the vertex v (as it is on a closed geodesic of length at most $\mathcal{N}'$).

If all the edges of T are non zero in $\pi_1(N,v)$, they correspond to three non trivial closed geodesics c_1, c_2 and c_3, starting and ending at the point v. In the universal cover $\widehat{N}$ of N, we can choose lifts a_1, a_2 and a_3 of c_1, c_2 and c_3 that bound a totally geodesic triangle $\mathbb{T}$. By definition, the covering projection maps a_i to c_i for $i=1,2,3$. The simplicial triangle T' corresponding to T is the image under the covering projection of the totally geodesic triangle $\mathbb{T}$ in $\widehat{N}$. As the covering projection is an isometry from the interior of $\mathbb{T}$ to the interior of T', and as each point in the interior of $\mathbb{T}$ lies at distance at most $\mathcal{N}'$ (which is an upper bound for the maximum of the lengths of the sides a_1, a_2 and a_3), each point x' in the interior of T' lies at distance at most $\mathcal{N}'$ from the vertex v.

Therefore, starting from the triangulated surface S, we can build a simplicial surface S' such that v is the only vertex of the simplicial structure of S' and each point x' in S' is at distance at most $\mathcal{N}'$ from v. In particular, the diameter of S' is at most $2\mathcal{N}'$. As the diameter of S is at most $\mathcal{N}$ and that v is also a point of S, for any points $x'\in S'$ and $x\in S$, we have :

$$\begin{aligned} d(x,x') &\leq d(x,v)+d(x',v)\\ &\leq \operatorname{diam}(S)+\mathcal{N}'\\ &\leq \mathcal{N}+\mathcal{N}', \end{aligned}$$

which proves lemma 13. □

Given a spine Γ for the compression body C_i which is a union of simplicial surfaces corresponding to ∂_-C_i joined by geodesic arcs, there exists a simplicial surface homotopic to this spine, by a homotopy that does not sweep out too much volume. More precisely, this follows from the following general lemma, proven in [Mah, Lemma 4.3 p. 2250].

Lemma 14. [Mah, Lemma 4.3]
Let $\sigma_1,\ldots,\sigma_n$ be a collection of simplicial surfaces, with basepoints v_i in N, a complete Riemannian 3-manifold of sectional curvature at most -1. Join the basepoint v_1 to each of the other basepoints by at least one geodesic arc to obtain a geodesic 2-complex Γ homotopic to a surface of genus g. Then, there exists a homotopy of Γ to a simplicial surface Σ_0 of genus g, and this homotopy sweeps out a volume of at most $3(2g+2)\Delta$, where Δ is the maximal volume of an ideal hyperbolic tetrahedron. □

Lemma 15. *Under assumptions (1) and (2), the modifications of the volume and diameter of C_i when replacing the boundary surfaces of C_i by simplicial surfaces as in the two lemma 13 and 14 become asymptotically negligible with respect to* $\operatorname{Vol}(C_i)$ *and* $\operatorname{diam}(C_i)$.

<u>Proof of lemma 15.</u>
First, let us denote by $\partial_-C_i=S_1\cup\ldots\cup S_n$ the components of ∂_-C_i, with $g(S_1)+\ldots+g(S_n)\leq g(\partial_+C_i)$. As in lemma 13, let us replace ∂_+C_i and $\partial_-C_i=S_1\cup\ldots\cup S_n$ by simplicial surfaces not too far away. The modification of the diameter of C_i is then at most

$$\begin{aligned} 4\epsilon\mathcal{N} &= 4\epsilon\mathcal{N}_i=4\epsilon a'(|\chi(\partial_+C_i)|+|\chi(S_1)|+\ldots+|\chi(S_n)|)\\ &= 4\epsilon a'\times 2(g(\partial_+C_i)+g(S_1)+\ldots+g(S_n)-1-n)\\ &\leq 4\epsilon a'\times 2(2g(\partial_+C_i)-2)\\ &\leq 8\epsilon a'\,|\chi(\partial_+C_i)|\,. \end{aligned}$$

As $|\chi(\partial_+ C_i)| \leq c_+(F_i)$, the modification of the diameter of C_i is at most $8\epsilon a' c_+(F_i)$. By inequality (3), $\operatorname{diam} C_i \geq \ln(d_i/(\chi_-^h(M_i)+2)) + \ln(2\mathrm{Vol}(M)/\pi)$. Hence

$$\frac{4\epsilon \mathcal{N}_i}{\operatorname{diam}(C_i)} \quad \leq \quad \frac{8\epsilon a' c_+(F_i)}{\ln(\frac{d_i}{\chi_-^h(M_i)+2}) + \ln(\frac{2\mathrm{Vol}(M)}{\pi})}.$$

By assumption (2), $\lim_{i\to+\infty} c_+(F_i)/\sqrt{\ln\ln\frac{d_i}{\chi_-^h(M_i)}} = 0$, so in particular

$$\lim_{i\to+\infty} \frac{8\epsilon a' c_+(F_i)}{\ln(\frac{d_i}{\chi_-^h(M_i)+2}) + \ln(\frac{2\mathrm{Vol}(M)}{\pi})} = 0,$$

and the modification of the diameter of C_i can then be neglected when i is large enough..

In lemma 14, the genus $g = g(C_i)$ is at most $c_+(F_i)/2+1$, so the volume swept out by the homotopy between Γ and Σ_0 is at most $3(c_+(F_i)+4)\Delta$. As by lemma 6, the volume of C_i is at least $\mathrm{Vol}(M)d_i/(\chi_-^h(M_i)+2)$, the volume of what remains after cutting the metric completion of C_i along Σ_0 and throwing off the component containing the infinite product regions is at least

$$\mathrm{Vol}(C_i) - 3(c_+(F_i)+4)\Delta \geq \mathrm{Vol}(C_i)\left(1 - 3\Delta(c_+(F_i)+4)\frac{\chi_-^h(M_i)+2}{\mathrm{Vol}(M)d_i}\right)$$

By assumption (2), $\lim_{i\to+\infty} c_+(F_i)/\sqrt{\ln\ln\frac{d_i}{\chi_-^h(M_i)}} = 0$, so in particular $\lim_{i\to+\infty} c_+(F_i)\chi_-^h(M_i)/d_i = 0$. By assumption (1), we have $\lim_{i\to+\infty} \chi_-^h(M_i)/d_i = 0$, and of course also $\lim_{i\to+\infty} c_+(F_i)/d_i = 0$. Finally,

$$\lim_{i\to+\infty} 3\Delta(c_+(F_i)+4)\frac{\chi_-^h(M_i)+2}{\mathrm{Vol}(M)d_i} = 0,$$

and this correction to the volume of C_i introduced at lemma 14 becomes negligible with respect to $\mathrm{Vol}(C_i)$ when i is large enough. □

In the sequel, we will consider the closure of the region of C_i bounded by the two connected simplicial surfaces, Σ_0 corresponding to the union of $\partial_- C_i$ and some arcs, and Σ_1 corresponding to $\partial_+ C_i$. As they can be neglected by lemma 15, we will forget about the variations of diameter and volume this modification introduces.

We have a sweepout in C_i such that Σ_0 and Σ_1 are sweepout surfaces. The following lemma ensures that we can homotop the sweepout between Σ_0 and Σ_1 to a simplicial sweepout. It is an improvement of [BCW, Theorem 2.3], and is proven by Maher [Mah, Lemma 2.5 p. 2236].

Lemma 16. [Mah, Lemma 2.5]
Let N be a closed orientable Riemannian manifold of sectional curvature at most -1. If Σ_0 and Σ_1 are simplicial surfaces with one-vertex triangulations, which are homotopic by a homotopy $\Phi : S \times I \to N$, then there exists a simplicial sweepout $\Phi' : S \times I \to N$ homotopic to Φ relative to $S \times \partial I$. □

Therefore, we can suppose that the sweepout in the compression body C_i is simplicial between the simplicial surfaces $\Sigma_0 = S_0$ and $\Sigma_1 = S_1$.

After getting this simplicial sweepout in the compression body C_i, the next step will be to get rid of the long and thin tubes in the sweepout surfaces to get a "generalised sweepout" in which the ϵ-diameter of all sweepout surfaces is uniformly bounded from above.

Definition 6. [Mah, Definition 3.2 p. 2237]
*Let N be a compact, connected and oriented 3-manifold. A **generalised sweepout** of N is given by a triple (Σ, f, h) where Σ is an orientable and compact 3-manifold, the map $h : \Sigma \to \mathbb{R}$ is a Morse function, constant on each boundary component of Σ and such that for all but finitely many $t \in \mathbb{R}$, the set $f^{-1}(\{t\})$ is an immersed surface. Moreover, it is required that $f : (\Sigma, \partial\Sigma) \to (N, \partial N)$ is of homological degree one.*

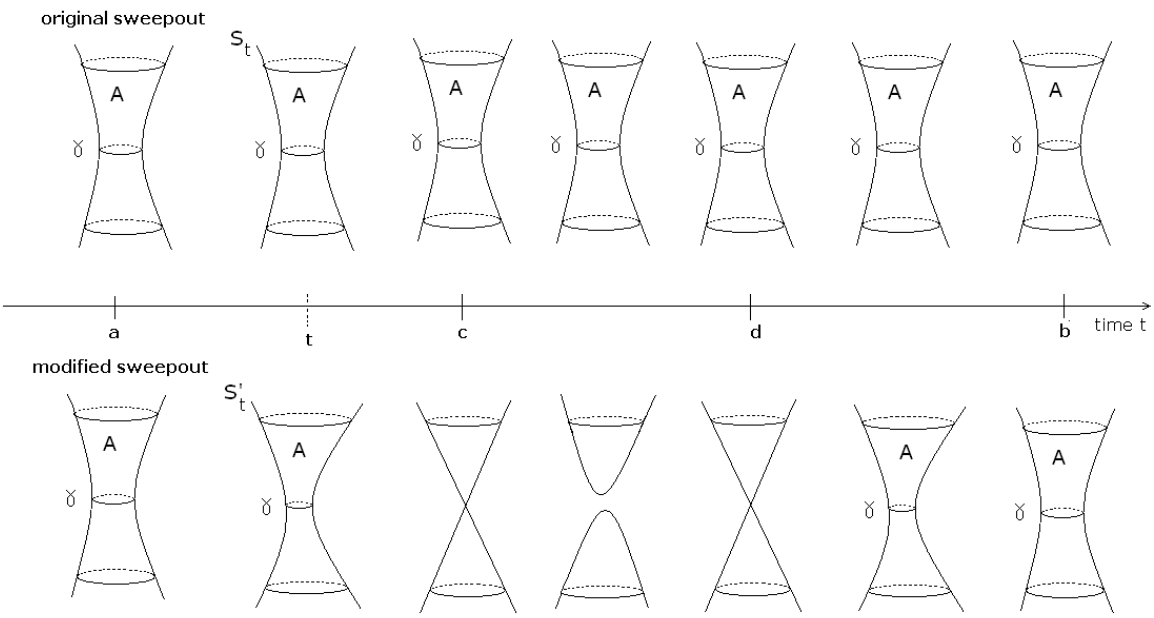
Of course, an ordinary sweepout $\Phi : S \times I \to N$ is a generalised sweepout : the Morse function $h : S \times I \to \mathbb{R}$ is given by the projection to the factor I, and for all $t \in (0,1)$, $h^{-1}(\{t\}) = S_t$ is

a surface in N. By definition of a sweepout, $\Phi : (S \times I, S \times \partial I) \to (N, \partial N)$ is of homological degree one.

For all $x \in \Sigma$, we think of $h(x) = t$ as the time coordinate. A generalised sweepout can be seen as a one-parameter family of immersed surfaces S_t with singular times t where the genus or the number of components of those surfaces change.

Starting from the simplicial sweepout $\{S_t\}_{t \in I}$ of C_i, we wish to obtain a generalised sweepout in which each sweepout surface has bounded ϵ-diameter. Here, we recall that ϵ is the injectivity radius of M, which provides a uniform lower bound for the injectivity radius of M_i. To this aim, we follow Maher and introduce the notion of **modification** of a sweepout.

Let (Σ, f, h) be a generalised sweepout of a 3-manifold N. Take a submanifold in Σ of the form $A \times [a, b]$ where $0 < a < b < 1$ and A is an annulus in the surfaces S_t for $t \in [a, b]$. We do $(0, 1)$ surgery to this solid torus $A \times [a, b]$ in the following way : choose two times c and d such that $a < c < d < b$. Take a chore geodesic γ for the annulus A in the surface S_a. Shrink this geodesic : it gets shorter and shorter, until it collapses to a point in a modification S'_c of the surface S_c. For all $t \in (c, d)$, replace the surfaces S_t by the surfaces S'_t obtained from S_t by surgering along γ, i.e. we cut S_t along γ and cap off the resulting surface with two discs. Do this in a differentiable way, such that the two discs of S'_t get closer and shrink to a single point at time d. The new surface S'_d is then singular, with a singular point corresponding to the two former discs. This point becomes again the geodesic γ that gets larger when $t \in (d, b]$ increases. Do this in such a way that you do not modify S_a nor S_b nor $\partial A \times [a, b]$. In this way, we get a new generalised sweepout (Σ', f', h'), where Σ' is obtained by replacing $A \times [a, b] \subset \Sigma$ by the new manifold where S_t is replaced by S'_t for all $t \in [a, b]$. Let us denote by T the small tube in N bounded by A, where the surgeries take place. The new maps (f', h') coincide with (f, h) outside $T \times [a, b]$ and in $\partial(T \times [a, b])$. As the modification of the sweepout takes place in a proper compact subset of N, there exists a point x in the interior of $N \setminus (T \times [a, b])$. As the map f is not modified in a neighborhood of $f^{-1}(\{x\})$, the homological degree of f' is the same as the homological degree of f, so it is still equal to one. Thus the triple (Σ', f', h') is still a generalised sweepout.



Definition 7. *The construction described above to obtain a new generalised sweepout* (Σ', f', h') *from the original generalised sweepout* (Σ, f, h) *is called a* ***modification of generalised sweepouts****. (In fact, this is a special case of a more general construction called a modification of generalised sweepouts, as described in* [MR]*.)*

Roughly, the idea is to cut along short curves in the simplicial surfaces S_t of length less than ϵ and to replace them by ruled discs, to get rid of too thin and long tubes. This is achieved by

Maher in the third section of [Mah, p. 2238 to p. 2245]. We recall here the proof, and we bring some precisions when they appear to be necessary.

Let t be a regular time. The simplicial surface S_t is composed of ruled triangles with at most one vertex of angle sum less than 2π, denoted by v_t. Let $\overline{S}_t$ be the completion of the universal cover $\widehat{S}_t$ of $S_t \setminus \{v_t\}$. As it is a metric 2-complex composed of triangles of curvature at most -1 and with vertices whose cone angles are all greater than or equal to 2π, $\overline{S}_t$ is a complete CAT(-1) geodesic metric space. Those spaces satisfy some useful properties, see [BH] and [Mah, p. 2239].

Let α be a homotopy class in $S_t \setminus \{v_t\}$. To α, we can associate a covering transformation of the universal cover of $S_t \setminus \{v_t\}$, which can be extended to an isometry of $\overline{S}_t$. As the completion of a fundamental domain for $S_t \setminus \{v_t\}$ is compact, this isometry cannot be parabolic. Thus it is hyperbolic or elliptic. Let $\overline{\gamma}_t$ be the set of points in $\overline{S}_t$ which are moved the least distance by the isometry. This is a geodesic if the isometry is hyperbolic, or isolated points if the isometry is elliptic. We denote by γ_t the projection of $\overline{\gamma}_t$ under the covering map, in the sense that if $\overline{\gamma}_t$ is a geodesic and does not meet $\overline{S}_t \setminus \widehat{S}_t$, γ_t is a closed piecewise geodesic homotopic to α in $S_t \setminus \{v_t\}$. If $\overline{\gamma}_t$ is a geodesic meeting $\overline{S}_t \setminus \widehat{S}_t$, then we perturb it slightly and in an equivariant way such that it avoids $\overline{S}_t \setminus \widehat{S}_t$ and its projection γ_t in $S_t \setminus \{v_t\}$ is an embedded closed curve in the homotopy class of α. Finally, if $\overline{\gamma}_t$ is a set of points, it corresponds to the constant loop γ_t of length zero and equal to the point v_t. By extension, in any case we will call γ_t **the geodesic representative of** α. Notice that γ_t is an embedded curve or a point in C_i.

As the negatively curved triangles that compose the surfaces S_t vary continuously with the time t, we can expect the geodesic representatives γ_t to vary also continuously. This is proven by Maher [Mah, Lemma 3.4 p. 2240].

Lemma 17. [Mah, Lemma 3.4]

Let γ be a simple closed curve in $S \setminus \{v\}$ where v is a point of S mapping to the point v_t for each time t. Then the geodesic representatives γ_t of γ vary continuously with t. □

We recall that ϵ is the injectivity radius of M, which is also a lower bound for the injectivity radius of M_i.

Definition 8. *A geodesic representative γ_t is said to be **short** if its length is less than or equal to ϵ.*

For all t, let Γ_t be the set of short geodesic representatives of S_t. This is a finite set and it is not empty, as the geodesic representative of the homotopy class of the loop around v_t has length zero.

Let γ_t be a short geodesic representative. Pick up a connected component $\widetilde{\gamma}_t$ of $\overline{\gamma}_t$, the preimage of γ_t in $\overline{S}_t$. Choose an orientation for $\widetilde{\gamma}_t$ so that the distance function from $\widetilde{\gamma}_t$ has a well defined sign. In the special case where the length of γ_t is zero, the distance from $\widetilde{\gamma}_t$ will always be non negative. If $[p, q]$ is an interval of $\mathbb{R}$, let $\widetilde{N}_{[p,q]}(\widetilde{\gamma}_t)$ be the set of points $x \in \overline{S}_t$ such that $p \leq d(x, \widetilde{\gamma}_t) \leq q$. Let $N_{[p,q]}(\gamma_t)$ be the image in S_t of $\widetilde{N}_{[p,q]}(\widetilde{\gamma}_t)$ under the covering projection. If the interval is the single point $\{r\}$, we will denote this neighborhood by $N_{[r]}(\gamma_t)$.

Definition 9. *Let $\mathcal{A}(\gamma_t)$ be the maximal neighborhood $N_{[p,q]}(\gamma_t)$ such that for every $r \in [p, q]$, $N_{[r]}(\gamma_t)$ is an embedded simple curve of length at most ϵ. We will call $\mathcal{A}(\gamma_t)$ the **annular neighborhood** of γ_t.*

*Let $\mathcal{E}(\gamma_t) = N_{[p+\epsilon/2,q-\epsilon/2]}(\gamma_t)$ be the **surgery neighborhood** corresponding to γ_t, with the convention that $\mathcal{E}(\gamma_t)$ is the empty set if $b - a < \epsilon$. This neighborhood is the subset of $\mathcal{A}(\gamma_t)$ corresponding to the union of all curves $N_{[r]}(\gamma_t)$ lying at distance at least $\frac{\epsilon}{2}$ from the boundary of $\mathcal{A}(\gamma_t)$.*

As the annular neighborhood $\mathcal{A}(\gamma_t)$ contains $\gamma_t = N_{[0]}(\gamma_t)$, it is not empty. The annular neighborhood and the surgery neighborhood vary continuously with t, but the surgery neighborhood $\mathcal{E}(\gamma_t)$ can be empty, and it does not necessarily contain the geodesic representative γ_t.

The following lemma is proven by Maher in [Mah, Lemma 3.7 p. 2242].

Lemma 18. [Mah, Lemma 3.7]

If α_t and β_t are short geodesic representative of distinct homotopy classes in $S_t \setminus v_t$, then their surgery neighborhoods $\mathcal{E}(\alpha_t)$ and $\mathcal{E}(\beta_t)$ are disjoint. □

We notice that this lemma implies that for each time t, there are at most $2g-1$ surgery neighborhoods, where g is the genus of the sweepout surface S_t.

This lemma shows that surgery neighborhoods are good "candidates" to do surgery on the sweepout surface S_t to get a generalized sweepout where the diameter of the thin tubes can be controlled. More precisely, the idea is to remove the surgery neighborhoods from the sweepout surfaces to get a new generalised sweepout $(\widehat{S}_t)_{t\in I}$. We describe this construction in details.

Let $\mathcal{E}(\gamma_t)$ be a surgery neighborhood, and $[a,b]$ a maximal time interval on which $\mathcal{E}(\gamma_t)$ is not empty. We have $0 \leq a \leq b \leq 1$. First, suppose that $0 < a \leq b < 1$, i.e. that $[a,b]$ is contained in the interior of I. Then $\mathcal{E}(\gamma_a)$ and $\mathcal{E}(\gamma_b)$ are two embedded simple curves $N_{[r_a]}(\gamma_a)$ and $N_{[r_b]}(\gamma_b)$, and the union of the surgery neighborhoods $\mathcal{E}(\gamma_{[a,b]}) = \{\mathcal{E}(\gamma_t)\,,\, t \in [a,b]\}$ is a solid torus in Σ, on which we wish to do a modification of generalised sweepouts. We follow Maher's construction [Mah, p. 2242 and 2243].

Choose a continuous family of basepoints on the boundary of $\mathcal{E}_t$ such that the two basepoints agree at times a and b. We modify the sweepout by expanding times a and b to short intervals I_a and I_b on which the map is constant for the moment.

On the interval I_a, the curve $N_{[r_a]}(\gamma_a)$ is an embedded simple curve of length less than or equal to ϵ. As the injectivity radius of the manifold M_i is at least ϵ, this curve is contained in an embedded ball B in M_i of radius $3\epsilon/4$ (as if we decide that the basepoint x_a of $N_{[r_a]}(\gamma_a)$ will be the center of the ball, each point of $N_{[r_a]}(\gamma_a)$ lies at distance at most $\epsilon/2$ from x_a). Therefore, the curve $N_{[r_a]}(\gamma_a)$ is nullhomotopic in M_i. It bounds a disc in M_i. In fact, we want to show that this curve bounds a disc in C_i.

Lemma 19. *Let c an embedded simple curve in one of the sweepout surfaces S_t of length less than or equal to ϵ. Then c bounds a disc in the compression body C_i.*

Proof of lemma 19.

If the curve c is nullhomotopic in C_i, it bounds a disc in C_i. So we can assume that c is an essential loop in C_i.

We have already seen that as the length of c is at most ϵ, it is contained in an embedded ball B in M_i of radius $3\epsilon/4$.

Let C_i' be the compression body in M_i adjacent to C_i along $\partial_+ C_i$. The boundary of the manifold $C_i \cup C_i'$ is $\partial_- C_i \cup \partial_- C_i'$ and the surface $\partial_+ C_i$ is a strongly irreducible Heegaard surface for $C_i \cup C_i'$. First, let us show that we can isotope the ball B to make it is disjoint from $\partial_- C_i \cup \partial_- C_i'$. As the curve c is contained in the interior of $C_i \cup C_i'$, we can do all the necessary isotopies without modifying B in a neighborhood of c. Among all such isotopies of B, suppose that B is a ball whose boundary sphere ∂B intersects $\partial_- C_i \cup \partial_- C_i'$ minimally. If this intersection is not empty, pick up an innermost curve γ in $\partial B \cap (\partial_- C_i \cup \partial_- C_i')$. This curve bounds a disc D in ∂B whose interior is disjoint from $\partial_- C_i \cup \partial_- C_i'$. As $\partial_- C_i \cup \partial_- C_i'$ is a union of incompressible surfaces and the curve γ is nullhomotopic in M_i, it bounds a disc D' in $\partial_- C_i \cup \partial_- C_i'$. We can isotope the ball B such that the disc D is replaced by D' pushed a little inside $C_i \cup C_i'$, and this isotopy strictly decreases the number of components of $\partial B \cap (\partial_- C_i \cup \partial_- C_i')$, which is a contradiction. Therefore, the ball B does not meet $\partial_- C_i \cup \partial_- C_i'$, and is entirely contained in the interior of $C_i \cup C_i'$.

Now, as c is an essential loop in C_i contained in an embedded ball B in $C_i \cup C_i'$, by a result of Frohman ([Fro, Lemma 1.1], see also [Lac2006, Proof of Proposition 3.2 p. 338]), the Heegaard surface $\partial_+ C_i$ is reducible in $C_i \cup C_i'$, which contradicts the fact that it is strongly irreducible, hence irreducible. Therefore, the loop c is nullhomotopic and bounds a disc in C_i. $\square$

We can now carry on Maher's construction. In the interval I_a, we replace in a continuous way the curve $N_{[r_a]}(\gamma_a) = \mathcal{E}(\gamma_a)$ by a pair of ruled discs in C_i coned from the basepoint x_a. In the interval (a,b), we remove the surgery neighborhood $\mathcal{E}(\gamma_t)$ and we replace it by a pair of ruled discs in C_i coned from the basepoints of the boundary of the surgery neighborhood. Finally, in the interval I_b we paste the discs together to come back to the original surface. This is a modification of a generalised sweepout as defined above.

The following lemma directly follows from Lemmas 3.8 to 3.10 and is proven in [Mah, p. 2243 to 2246].

Lemma 20. *Let $\widehat{S}_t$ be the surface obtained from the simplicial sweepout surface S_t by replacing all surgery neighborhood $\mathcal{E}(\gamma_t)$ of S_t by pairs of ruled discs as described above. Then the ϵ-diameter of $\widehat{S}_t$ is at most $K_i := 4\left(3 + 1/\sinh^2(\epsilon/8)\right) g(C_i) - 10$.* □

Maher's construction does not take the boundaries into account. However, it may happen that $a = 0$ or $b = 1$, and in this case we might be obliged to modify the starting and finishing simplicial sweepout surfaces $S_0 = \Sigma_0$ and $S_1 = \Sigma_1$, which we want to avoid. Therefore, if this case occurs, we need to refine the construction to modify the simplicial sweepout in a small regular neighborhood of $S_0 \cup S_1$ in such a way that we do not modify the surfaces S_0 and S_1. As we will lose control on the ϵ-diameter of the sweepout surfaces in this regular neighborhood, we have to choose it small enough in order that the sweepout surfaces we will pick up later to be our nested surfaces are not in this neighborhood, to control their ϵ-diameter well thanks to lemma 20.

Before considering the case when $a = 0$ or $b = 1$, we need another lemma which will lead us to determine precisely the size of the neighborhoods of S_0. Let us denote by $K_i' := 2a' \,|\chi(\partial_+ C_i)|$. The number $2\epsilon K_i'$ is is an upper bound for the diameter of the simplicial surface Σ_1, which we will now identify to $\partial_+ C_i$. Let us denote the diameter of the compression body C_i by δ_i.

Lemma 21. *There exists a point x_0 in C_i in the interior of the region bounded by Σ_0 and Σ_1, and at distance at least $(\frac{\delta_i}{2} - \epsilon K_i')$ to $\partial_+ C_i$.*

Proof of lemma 21.

Suppose that lemma 21 is false. Let $\widetilde{C_i}$ be the closure of the region bounded by Σ_0 and Σ_1. We have seen that the diameter of $\widetilde{C_i}$ is asymptotically close to the diameter of C_i. So we will still denote by δ_i the diameter of $\widetilde{C_i}$. For every point z in $\widetilde{C_i}$, we would have $\mathrm{dist}(z, \partial_+ C_i) < \frac{\delta_i}{2} - \epsilon K_i'$. Take two points x and y in $\widetilde{C_i}$ such that $d(x, y) = \mathrm{diam}(\widetilde{C_i}) = \delta_i$. We have then :

$$
\begin{aligned}
d(x,y) = \delta_i &\leq \mathrm{dist}(x, \partial_+ C_i) + \mathrm{diam}(\partial_+ C_i) + \mathrm{dist}(y, \partial_+ C_i) \\
&< (\frac{\delta_i}{2} - \epsilon K_i') + 2\epsilon K_i' + (\frac{\delta_i}{2} - \epsilon K_i') \\
&< \delta_i,
\end{aligned}
$$

which provides a contradiction, establishing lemma 21. □

Let c be a length-minimizing geodesic arc from x_0 to $\partial_+ C_i$. Let us denote by μ the distance between the geodesic c and Σ_0. As the geodesic c lies in the interior of $\widetilde{C_i}$ (except one extremity lying on $\partial_+ C_i$), the number μ is strictly positive.

To finish to modify the original simplicial sweepout to get the desired generalized sweepout, there remains to consider the case when $a = 0$ or $b = 1$. If $\mathcal{E}(\gamma_0)$ is a single closed curve, we can apply the previous construction, replacing the time 0 by an interval I_0 and doing surgery on this interval, without modifying the starting boundary surface $S_0 = \Sigma_0$. It works similarly if $\mathcal{E}(\gamma_1)$ is a single closed curve. The problem is when $\mathcal{E}(\gamma_0)$ or $\mathcal{E}(\gamma_1)$ have non empty interior. As the two cases are similar, let us suppose for instance that there exists a maximal time $b \in (0, 1]$ such that $\mathcal{E}(\gamma_t)$ is a non empty surgery neighborhood for all $t \in [0, b]$ and that the interior of $\mathcal{E}(\gamma_0)$ is not empty.

As the sweepout surfaces $(S_t)_{t\in I}$ vary continuously with t, there exists a constant $\eta > 0$ as small as we like, depending only on the original simplicial sweepout $(S_t)_{t\in I}$ and the choice of the point x_0 and the geodesic arc c, such that for every $t \in [0, \eta]$, each point of S_t lies at distance at most $\mu/2$ from $\Sigma_0 = S_0$, and that for every $t \in [1-\eta, 1]$, each point in S_t is at distance at most $\epsilon K_i'/2$ from $\Sigma_1 = S_1$. If $b \leq \eta$, we do not modify the sweepout. Otherwise, if $\eta < b < 1$, we apply Maher's construction for all $t \in [\eta, b]$: we replace the surgery neighborhoods $\mathcal{E}(\gamma_t)$ by a pair of ruled discs coned from basepoints in the boundary of $\mathcal{E}(\gamma_t)$ in a continuous way. In the interval $[0, \eta]$, we replace the surgery neighborhoods by a pair of discs for t near η, that get pasted to the initial surgery neighborhood $\mathcal{E}(\gamma_0)$ as the time is decreasing to 0, not too far from the original surface S_t and in a continuous way. We can do this in such a way that it is still a modification of a generalised sweepout. If $b = 1$, do the same for all $t \in [1-\eta, 1]$. As the diameter of the ruled

discs is less than ϵ and $K_i'/2 \geq 1$, we can suppose that every point in $\widehat{S}_t$ is at distance at most $\epsilon K_i'$ from Σ_1 for all $t \in [1-\eta, 1]$.

We have thus proven the following lemma.

Lemma 22. *There exists a constant $\eta > 0$ as small as wanted, depending only on the simplicial sweepout $\{S_t\}_{t\in I}$ and the choice of the geodesic γ, and a finite sequence of modifications of the simplicial sweepout that gives a generalised sweepout $\{\widehat{S}_t\}_{t\in I}$ of C_i such that for every regular time $t \in [\eta, 1-\eta]$, the ϵ-diameter of $\widehat{S}_t$ is less than or equal to $K_i := 4\left(3 + 1/\sinh^2(\epsilon/8)\right) g(C_i) - 10$, and that any point in a surface $\widetilde{S}_t$ lies at distance at most $\epsilon K_i'$ from Σ_1 if $t \geq 1-\eta$, where $K_i' := 2a' |\chi(\partial_+ C_i)|$. Any point in one of the original sweepout surfaces S_t for $t \leq \eta$ lies at distance at most $\mu/2$ from Σ_0. Moreover, for every regular time t, the surface $\widehat{S}_t$ is homotopic to an embedded surface obtained from $\partial_+ C_i$ by surgery.*

Now, for completeness of the proof of proposition 7, we state and prove a few lemmas which are implicit in [Mah, proof of Lemma 4.5 p. 2251].

We recall from definition 3 that if x is a point in C_i and S an immersed surface of C_i, we say that S **separates** x **from** $\partial_+ C_i$ if every oriented path from x to $\partial_+ C_i$ has its algebraic intersection number equal to $+1$.

If two surfaces S and T immersed in C_i are such that S separates every point of T from $\partial_+ C_i$, we say that T **separates** S **from** $\partial_+ C_i$. In this case, the surfaces S and T are said to be **nested**.

Lemma 23. *A point x in the interior of C_i is separated from $\partial_+ C_i$ by $\widehat{S}_t$ if and only if there exists a path γ from x to $\partial_+ C_i$ with algebraic intersection number with $\widehat{S}_t$ equal to $+1$.*

Proof of lemma 23.

It suffices to show that if there exists a path γ from x to $\partial_+ C_i$ with algebraic intersection number with $\widehat{S}_t$ equal to $+1$, then every path γ' from x to $\partial_+ C_i$ has its algebraic intersection number with $\widehat{S}_t$ equal to $+1$.

Let γ' be another path from x to $\partial_+ C_i$ in C_i. As the immersed surface $\widehat{S}_t$ is homologous to $\partial_- C_i$, the homology class $[\widehat{S}_t]$ is equal to zero in $H_2(C_i, \partial_- C_i)$. The composition $\alpha = \gamma^{-1} \cdot \gamma'$ is a 1-cycle in $H_1(C_i, \partial_+ C_i)$.

As $\partial C_i = \partial_- C_i \cup \partial_+ C_i$, we have $[\alpha] \cdot [\widehat{S}_t] = [\alpha] \cdot 0 = 0$, so $\gamma' \cdot \widehat{S}_t = \gamma \cdot \widehat{S}_t = +1$, which proves lemma 23. $\square$

For all $t \in [0,1]$, let D_t be the closure of the set of points $x \in C_i$ separated from $\partial_+ C_i$ by $\widehat{S}_t$. As the immersed surfaces $\left(\widehat{S}_t\right)_{t\in[0,1]}$ are generalised sweepout surfaces of the compression body C_i, D_0 is the starting sweepout surface $\widehat{S}_0$, and D_1 is equal to the whole compression body C_i. Let E_t be the component of D_t containing x_0. As before, E_0 is a complex of dimension at most 2, and $E_1 = C_i$.

Lemma 24. *The boundary of the set D_t is the surface $\widehat{S}_t$.*

Proof of lemma 24.

Let x be a point in $\widehat{S}_t$. As $\widehat{S}_t$ is a generalised sweepout surface for the compression body C_i, there exists a path c from x to $\partial_+ C_i$ such that for every point y of c distinct from x, the path $c_{|y}$ obtained from c by deleting the interval $[x, y)$ does not intersect the surface $\widehat{S}_t$. So in particular, for every point y on c distinct from x, the algebraic intersection number between $c_{|y}$ and $\widehat{S}_t$ is zero, and y is in the complement of D_t in C_i. As the point x is a limit of such points y, x is in the closure of the complement of D_t in C_i. But as x is on the surface $\widehat{S}_t$ and that this surface separates the compression body C_i, every point close enough to x and on the other side of $\widehat{S}_t$ with respect to y is separated by $\widehat{S}_t$ from $\partial_+ C_i$. As the set D_t is closed and that x is a limit of points of D_t, the point x also belongs to D_t. Therefore, the point x lies in the boundary of D_t, and the surface $\widehat{S}_t$ is a subset of the boundary of D_t.

To get the reverse inclusion, let us assume that there exists a point x in the boundary of D_t which does not belong to the surface $\widehat{S}_t$, and seek for a contradiction. The distance $d = \mathrm{dist}(x, \widehat{S}_t)$

is then strictly positive. As the point x belongs to the boundary of D_t, there exists a point y in the complement of D_t in C_i such that $d(x,y) \leq \frac{d}{2}$. As y is in the complement of D_t, there is a path c from y to the boundary $\partial_+ C_i$ with algebraic intersection number with $\widehat{S_t}$ different from $+1$. Let c' be a minimizing geodesic from x to y : as the length of c', which is equal to the distance between x and y, is strictly less than the distance of x to $\widehat{S_t}$, the geodesic c' does not intersect the surface $\widehat{S_t}$. If $c'' = c' \cup c$, c'' is a path from x to $\partial_+ C_i$ with algebraic intersection number with $\widehat{S_t}$ not equal to $+1$. Therefore, the point x is not separated from $\partial_+ C_i$ by $\widehat{S_t}$.

But as the point x belongs also to D_t, there exists a point z in C_i separated from $\partial_+ C_i$ by $\widehat{S_t}$ and such that the distance between z and x is less than $\frac{d}{2}$. Take a minimizing geodesic a from z to x. Let us denote by $b = a \cup c''$. The path b is linking z to $\partial_+ C_i$, which implies that the algebraic intersection number of b with $\widehat{S_t}$ is equal to $+1$. From the other hand, the distance between z and x is at most $\frac{d}{2} < \mathrm{dist}(x, \widehat{S_t})$, which implies that the minimizing geodesic a does not intersect the surface $\widehat{S_t}$. But then, the algebraic intersection number of the path $b = a \cup c''$ with the surface $\widehat{S_t}$ is not equal to $+1$, which contradicts the fact that z is separated from $\partial_+ C_i$ by $\widehat{S_t}$. Thus, the point x necessarily belongs to the surface $\widehat{S_t}$, which ends the proof of lemma 24. □

Lemma 25. *For every time t, the boundary of E_t is connected.*

Proof of lemma 25.

Indeed, if the boundary of E_t would not be connected, it would have at least two components S and T of $\widehat{S_t}$. But then, S and T would be two disjoint and separating surfaces in the compression body C_i. If they are not nested, the set of the points separated from $\partial_+ C_i$ by S is disjoint to the set of points separated from $\partial_+ C_i$ by T, which contradicts the fact that E_t is connected. Therefore, the surfaces S and T are nested. For example, let us assume that S separates T from $\partial_+ C_i$. This means that for each point x in T, any path from x to $\partial_+ C_i$ has its intersection number with S equal to $+1$. Thus, this path will intersect $\widehat{S_t}$ again. This is in contradiction with the fact that $\widehat{S_t}$ is a generalised sweepout surface : if we take the path which is the track of x in the sweepout, this path intersects $\widehat{S_t}$ only once. □

To prove proposition 7, we will pick up the desired nested surfaces among the family of connected surfaces $(\partial E_t)_{t \in [0,1]}$.

End of the proof of proposition 7.

Let c be the length-minimizing geodesic arc from x_0 to $\partial_+ C_i$. Let L be the length of c. From lemma 21, we have $L \geq \frac{\delta_i}{2} - \epsilon K'_i$. Take $\ell \mapsto c(\ell)$ an arc-length parameterization of c, such that $c(0) = x_0$ and $c(L) = y_0 \in \partial_+ C_i$.

First, let us show that $E_t = \emptyset$ for $t \in [0, \eta]$. As every original sweepout surface S_t is contained in a $\mu/2$-neighborhood of $S_0 = \widehat{S_0}$ for all $t \leq \eta$, and that the distance between c and S_0 is at least μ, the geodesic c does not meet the sweepout surfaces S_t for every $t \leq \eta$. As the new sweepout surfaces $\widehat{S_t}$ are obtained from the surfaces S_t by surgery, the intersection number between c and $\widehat{S_t}$ is the same as the intersection number between c and S_t, so it is zero for $t \leq \eta$. Therefore, the geodesic c is an arc joining x_0 to $\partial_+ C_i$ with intersection number with $\widehat{S_t}$ equal to zero for $t \leq \eta$. By definition, the surfaces $\widehat{S_t}$ do not separate x_0 from $\partial_+ C_i$ for $t \leq \eta$, showing that there is no component of D_t containing x_0. Thus, $E_t = \emptyset$ for every $t \in [0, \eta]$.

Let us assume that $\frac{\delta_i}{2} - 2\epsilon K'_i \geq 5\epsilon K_i$. As the sets E_t vary continuously with the time t, the function $\mathcal{L}$ which maps the time t to the length of $c \cap E_t$ is a continuous map. From the fact that $\mathcal{L}(\eta) = 0$ and $\mathcal{L}(1) = L$ the length of c, we deduce that there is a time $t_1 \in (\eta, 1)$ such that $\mathcal{L}(t_1) = L - 2\epsilon K_i - \epsilon K'_i$. Let S_1 be the boundary of E_{t_1}. From lemma 25, we deduce that the immersed surface S_1 is a connected component of $\widehat{S_{t_1}}$. As c is a minimizing arc-length parametrized geodesic, for every a and $b \in [0, L]$, we have $d(c(a), c(b)) = |b - a|$. Thus, the intersection point $c(\mathcal{L}(t_1))$ between S_1 and c is lying at distance $2\epsilon K_i + \epsilon K'_i$ from $\partial_+ C_i$. Since by construction every point in the surface $\widehat{S_t}$ for $t \geq 1 - \eta$ is at distance at most $\epsilon K'_i$ from $\partial_+ C_i$,

necessarily $t_1 < 1-\eta$. As the sets $E_t = \emptyset$ for $t \leq \eta$, in fact $\eta < t_1 < 1-\eta$. By definition of E_{t_1}, the surface S_1 separates x_0 from $\partial_+ C_i$. As S_1 is connected, by lemma 22 its diameter is at most $2\epsilon K_i$. Therefore, the surface S_1 cannot meet $\{c(\ell)\,,\, 0 \leq \ell < L-4\epsilon K_i - \epsilon K_i')\} \cup \{c(\ell)\,,\, L-\epsilon K_i' < \ell \leq L\}$. Let ℓ_1 be the smallest value of ℓ such that $c(\ell) \in S_1$. We have $L - 4\epsilon K_i - \epsilon K_i' \leq \ell_1 \leq L - \epsilon K_i'$, with $L - 4\epsilon K_i - \epsilon K_i' \geq \frac{\delta_i}{2} - 4\epsilon K_i - 2\epsilon K_i' \geq \epsilon K_i > 0$.

Let $c_1 = \{c(\ell)\,,\, 0 \leq \ell \leq \ell_1 - 14\epsilon K_i\}$. Replacing c by c_1, we can iterate the previous process. If K_i is small enough compared to δ_i, there exists a time t_2 such that the length of $c_1 \cap E_{t_2}$ is equal to : length$\,(c_1) - 2\epsilon K_i = \ell_1 - 16\epsilon K_i \geq L - 20\epsilon K_i - \epsilon K_i'$. For the same reasons as before, the boundary of E_{t_2} is a surface S_2 which is a connected component of $\widehat{S}_{t_2}$ separating x_0 from $\partial_+ C_i$, and it intersects c_1 only on the set $\{c_1(\ell)\,,\, (\ell_1 - 14\epsilon K_i) - 4\epsilon K_i \leq \ell \leq \ell_1 - 14\epsilon K_i\}$.

Let us prove that the distance between the surfaces S_1 and S_2 is less than or equal to $10\epsilon K_i$. Let ℓ_2 be the smallest real number ℓ such that $c(\ell) \in S_2$. From the former discussion, $\ell_2 \leq \ell_1 - 14\epsilon K_i$. As $c(\ell_1) \in S_1$ and $c(\ell_2) \in S_2$, we have :

$$\begin{aligned}
\mathrm{dist}(S_1, S_2) &\geq \mathrm{dist}(c(\ell_1), c(\ell_2)) - \mathrm{diam}(S_1) - \mathrm{diam}(S_2) \\
&\geq (\ell_1 - \ell_2) - 4\epsilon K_i \\
&\geq 14\epsilon K_i - 4\epsilon K_i = 10\epsilon K_i.
\end{aligned}$$

We can iterate the process with $c_2 = \{c(\ell)\,,\, 0 \leq \ell \leq \ell_2 - 14\epsilon K_i\}$, on condition that $\ell_2 - 14\epsilon K_i > 4\epsilon K_i$, so for example if $L - 2 \times 18\epsilon K_i - \epsilon K_i' > 4\epsilon K_i$.

The iterations stop when $L - 18\epsilon K_i(n_i - 1) - \epsilon K_i' > 4\epsilon K_i$ but $L - 18\epsilon K_i n_i - \epsilon K_i' \leq 4\epsilon K_i$, so for $n_i = \lceil \frac{L - \epsilon(4K_i + K_i')}{18\epsilon K_i} \rceil$. As $L \geq \frac{\delta_i}{2} - \epsilon K_i'$, $n_i \geq \lceil \frac{\delta_i}{36\epsilon K_i} - \frac{2}{9} - \frac{K_i'}{9K_i} \rceil$, which proves proposition 7. $\square$

4. From nested to parallel surfaces.

We then prove proposition 8.

With proposition 7, we know that we can find $n_i = \lceil \frac{\delta_i}{36\epsilon K_i} - \frac{2}{9} - \frac{K_i'}{9K_i} \rceil$ immersed surfaces in the compression body C_i of the cover M_i. All those surfaces are nested, their diameter is at most $2\epsilon K_i$ and they are at distance at least $10\epsilon K_i$ from each other, where $K_i = 4\left(3 + 1/\sinh^2(\epsilon/8)\right) g(C_i) - 10$. Furthermore, all those surfaces are homotopic to embedded surfaces obtained from $\partial_+ C_i$ by surgery.

Thus the genus of those immersed surfaces is between 0 and $g(C_i) = g(\partial_+ C_i)$. So there are at least $n_i' = \lfloor n_i/(g(C_i) + 1) \rfloor$ surfaces $S_1, \ldots, S_{n_i'}$ with the same genus, which is bounded from above by $g(C_i)$. We take the indices j such that S_{j+1} separates S_j from $\partial_+ C_i$.

We then follow the proof of Maher [Mah, p. 2252 to p. 2257]. Let S be one of the previous immersed and nested surfaces with the same genus. A **collection of compression discs for** S is a finite set Δ of discs, properly embedded in C_i and such that the sweepout gives a homotopy from S to $\partial_+ C_i \cup \Delta$. The first step is to show that for two connected and nested sweepout surfaces, we can choose collections of compression discs such that one of them is a subset of the other one. This is done in [Mah, Lemma 4.6 p. 2252]. In particular, if the two surfaces have the same genus, they are homotopic.

Lemma 26. [Mah, Lemma 4.6]

Let S_1 and S_2 be two of the immersed surfaces obtained in proposition 7. Suppose for example that S_2 separates S_1 from $\partial_+ C_i$. Then we can choose a collection of compression discs Δ_1 for S_1 and Δ_2 for S_2 such that Δ_2 is a subset of Δ_1. $\square$

This lemma shows that all the nested surfaces $S_1, \ldots, S_{n_i'}$ are homotopic, as they have the same genus.

The following lemma is crucial : we whish to replace the nested immersed surfaces by embedded surfaces of the same genus in an arbitrarilly small neighborhood of the original immersed surfaces. This lemma is proven in [Mah, Lemma 4.7 p. 2253].

Lemma 27. [Mah, Lemma 4.7]

Let S be one of the surfaces obtained in proposition 7. Let T be a least genus connected embedded surface separating S from $\partial_+ C_i$. Then T is incompressible in $C_i \setminus S$ and the genus of T is greater than or equal to the genus of S.

Proof of lemma 27.

We recall here Maher's proof.

If the surface T were compressible in $C_i \setminus S$, it could be compressed along embedded discs in $C_i \setminus S$ to obtain a new surface T' embedded in $C_i \setminus S$. But one component of T' would be an embedded surface in C_i separating S from $\partial_+ C_i$, with genus strictly less than the genus of T, which is a contradiction. So the surface T is incompressible in $C_i \setminus S$.

The surface S is homotopic to $\partial_+ C_i$ compressed along a collection D of embedded discs. Thus $C_i' := C_i \setminus D$ is a compression body and we can find a spine Γ for C_i' that is homotopic to the immersed surface S. The map on first homology $H_1(\Gamma) \to H_1(C_i)$ induced by the inclusion of Γ in C_i is injective.

The surface T is an embedded surface in the compression body C_i, so it is separating and there exists a set Δ of embedded compression discs for T such that T compressed along Δ is parallel to some components of $\partial_- C_i$. As T is incompressible in $C_i \setminus S$, the compression discs of Δ for the surface T are only in one side of T. So the surface T bounds a compression body C_i'' in C_i. As the composition of the maps induced by the inclusions $H_1(\Gamma) \to H_1(C_i'') \to H_1(C_i)$ is injective, the map $H_1(\Gamma) \to H_1(C_i'')$ is injective. Thus the rank of $H_1(C_i'')$ is greater than or equal to the rank of $H_1(\Gamma)$, and necessarily the genus of T is greater than or equal to the genus of S. $\square$

A consequence of lemma 26 is that all the nested and immersed surfaces $S_1, \ldots, S_{n_i'}$ are homotopic. We want a little more : we need to find for all j between 1 and $(n_i' - 1)$ a homotopy between S_j and $S_{n_i'}$ that is disjoint from S_k for all $k < j$. We follow the arguments of the proof of [Mah, Lemma 4.8 p. 2254], bringing the precise bounds.

Lemma 28. *From the surfaces $S_1, \ldots, S_{n_i'}$, one can construct a collection of connected surfaces $S_1', \ldots, S_{n_i'-1}', S_{n_i'}'$ which are disjoint, nested and homotopic, and the homotopy from $S_{n_i'}$ to S_j' is disjoint from S_k' for $1 \leq k < j$. Furthermore, the diameter of the surfaces S_j' is at most $8\epsilon K_i$ and they are at distance at least $2\epsilon K_i$ from each other.*

Proof of lemma 28

Each surface S_j admits a one-vertex triangulation with edge-length bounded by $4\epsilon K_i$, and its diameter is at most $2\epsilon K_i$. Therefore, by lemma 13 the surfaces S_1 and $S_{n_i'}$ are homotopic to simplicial surfaces S_1' and $S_{n_i'}'$ with diameter at most $4\epsilon K_i$ and such that for every points $x \in S_j$ and $x' \in S_j'$ (where $j = 1$ and n_i'), the distance between x and x' is at most $6\epsilon K_i$. In fact, by construction of S_j', each point of S_j' is at distance at most $4\epsilon K_i$ from the original surface S_j.

The homotopy between the two simplicial surfaces S_1' and $S_{n_i'}'$ can be modified into a simplicial sweepout as in section 3. By lemma 22, there exists a modification of this simplicial sweepout where all the sweepout surfaces S_t' for $t \in [\eta, 1-\eta]$ have ϵ-diameter bounded above by K_i. We can use the same constant K_i as before since the genus of the surfaces S_j is at most $g(C_i)$. Moreover, the surfaces S_t' are homotopic to the surface S_n after some compressions if necessary. For j between 2 and $(n_i' - 1)$, let S_j' be the first sweepout surface S_t' intersecting S_j.

We know from the construction of a generalised sweepout that the genus of the surface S_j' is at most the genus of the surface S_j. In fact, we will show below that those two genera are equal.

Claim . *For all $1 \leq j \leq n_i' - 1$, the genus of the surface S_j' is the same as the genus of the original sweepout surface S_j.*

Asuming the claim, since the modified sweepout surfaces S_j' have the same genus as the origninal sweepout surfaces S_j, in fact there is no compression to obtain the surfaces S_j' and they were already sweepout surfaces of the original simplicial sweepout between S_1' and $S_{n_i'}'$. So the surfaces S_j' are homotopic to the surface $S_{n_i'}'$, and by definition of a sweepout, this homotopy is disjoint from the surfaces S_k' for every $k < j$.

Proof of claim.

Suppose that there exists some j such that the genus of S'_j is strictly less than the genus of S_j. By a result of Gabai, we can then replace our simplicial surface S'_j by an embedded surface T'_j in an arbitrarily small neighborhood of the immersed surface S'_j. More precisely, take a small regular neighborhood $N(S'_j)$ of the immersed surface S'_j. This neighborhood contains embedded surfaces in the same homology class as S'_j in $H_2(N(S'_j), \partial N(S'_j))$. Gabai showed that the singular norm on homology is the same as the embedded Thurston's norm [Ga], hence there exists an embedded surface T'_j in $N(S'_j)$ with the same homology class as S'_j and of genus less than or equal to the genus of S'_j. If we choose sufficiently small neighborhoods $N(S'_j)$, we can ensure that the diameter of the embedded surface T'_j is less than $3\epsilon K_i$. In particular, as the surfaces S'_1 and $S'_{n'_i}$ are too far away, the embedded surface T'_j is disjoint from S'_1 and $S'_{n'_i}$, and it is separating S'_1 from $S'_{n'_i}$. Applying lemma 27, we see that the genus of T'_j must be at least the genus of S'_1. But as the genus of S'_1 is the same as the genus of S_j, and that the genus of T'_j is at most the genus of S'_j, which we have supposed strictly less than the genus of S_j, we have $g(T'_j) < g(S'_1)$, which is a contradiction. □

As the surfaces S_j were at distance at least $10\epsilon K_i$ from each other and that every point of S'_j is at distance at most $4\epsilon K_i$ from the original surface S_j for all $j = 1, \dots, n'_i$, the new surfaces S'_j are at distace at most $2\epsilon K_i$ from each other (which also shows that the surfaces S'_j are all disjoint). Furthermore, their diameter is bounded from above by $8\epsilon K_i$.

There remains to show that the surfaces $S'_1, \dots, S'_{n'_i}$ are nested. In the spirit of the proof of proposition 7, let us denote by $D_{n'_i}$ the closure of the subset of the points of C_i separated from $\partial_+ C_i$ by $S'_{n'_i}$. For all $j < n'_i$, the surface S'_j intersects the surface S_j, which lies in $D_{n'_i}$. As S'_j is at distance at least $2\epsilon K_i$ from $S_{n'_i} = \partial D_{n'_i}$, S'_j is contained in the interior of $D_{n'_i}$. So it is separated from $\partial_+ C_i$ by $S'_{n'_i}$. Therefore, if we denote by D_j the closure of the points of C_i separated from $\partial_+ C_i$ by S'_j, $D_j \subset D_{n'_i}$. Let $1 \le k < j < n'_i$. If we take a point x in D_k, as $D_k \subset D_{n'_i}$, every path γ from x to $\partial_+ C_i$ has its algebraic intersection number with $\partial_+ C_i$ equal to $+1$. As the surface S'_j is homotopic to $S'_{n'_i}$ by a homotopy that is disjoint from S'_k, this homotopy does not change the intersection number, so the intersection number of γ with S'_j is still equal to $+1$, and x is in D_j. Thus $D_k \subset D_j$ for $1 \le k < j \le n'_i$, showing that the surfaces $S'_1, \dots, S'_{n'_i}$ are nested. This ends the proof of lemma 28. □

In the sequel, we replace the family $S_1, \dots, S_{n'_i}$ by the new family $S'_1, \dots, S'_{n'_i-1}, S'_{n'_i}$ of surfaces obtained by lemma 28, and for simplicity, we will still denote this family by $S_1, \dots, S_{n'_i}$.

We then wish to replace our immersed surfaces by embedded surfaces in an arbitrarily small neighborhood of the immersed surfaces. Take a small regular neighborhood $N(S_j)$ of one of the immersed and nested surfaces S_j. As in the proof of the claim, by Gabai [Ga], this neighborhood contains an embedded surface T_j in the same homology class as S_j in $H_2(N(S_j), \partial N(S_j))$ and of genus less than or equal to the genus of S_j. If we choose sufficiently small neighborhoods $N(S_j)$, we can ensure that the diameter of the embedded surfaces T_j is less than $9\epsilon K_i$, and two embedded surfaces T_j and T_k are at distance at least ϵK_i. The genus of T_j is at most the genus of S_j, but we wish to show that in fact, the genus of T_j is the same as the genus of S_j.

With lemma 27, we know that the genus of the embedded surface T_j for $j = 2, \dots, n'_i$ is greater than or equal to the genus of the immersed surface S_1 that it separates from $\partial_+ C_i$. But as the genus of T_j is at least the genus of S_j, which is equal to the genus of S_1, in fact the genus of T_j is equal to the genus of S_j : the surfaces $T_2, \dots, T_{n'_i}$ have the same genus as the immersed surfaces $S_2, \dots, S_{n'_i}$. The final step in the proof of proposition 8 is to show that some of the embedded surfaces are actually parallel.

Lemma 29. *The embedded surfaces $T_4, \dots, T_{n'_i-1}$ are parallel.*

<u>Proof of lemma 29.</u>

This lemma relies on homological arguments, see [Mah, Lemmas 4.9 to 4.11]. For completeness, we give here a shorter proof, based on classical 3-manifold topological results.

Let V be the 3-complex in C_i bounded by the immersed surfaces S_1 and $S_{n'_i}$. There is a sweepout ϕ between S_1 and $S_{n'_i}$ such that for each $1 \leq j \leq n'_i$, the surface S_j is a sweepout surface. In other words, the application induced by the map $\phi : S \times I \to V$ in homology $\phi_* : H_3(S \times I, \partial(S \times I)) \to H_3(V, \partial V)$ is an isomorphism and for each j, there exists a time $t_j \in I$ such that $S_j = \phi(S \times \{t_j\})$. Moreover, we have $0 = t_1 < t_2 < \ldots < t_{n'_i} = 1$.

By a classical construction (see [Sta, point 3. p. 96] for example), we can homotop the sweepout ϕ to a map ϕ' which is still degree one, and such that for every $2 \leq j \leq n'_i$, $\phi'^{-1}(T_j)$ is an embedded incompressible surface (not necessarily connected) in $S \times I$.

Take $3 < j < k \leq n'_i - 1$. As the homology class of the surfaces T_j and T_k is the same as the homology class of S_3, the homology class of the preimages $\phi'^{-1}(T_j)$ and $\phi'^{-1}(T_k)$ in $H_2(S \times [t_3, 1], \partial(S \times [t_3, 1]))$ is the same as the homology class of the fiber $S \times \{t\}$. As those preimages are incompressible embedded surfaces, $\phi'^{-1}(T_j)$ and $\phi'^{-1}(T_k)$ are each composed of an odd number of connected surfaces isotopic to the fiber $S \times \{t\}$ with total algebraic intersection number with any path from $S \times \{t_3\}$ to $S \times \{1\}$ equal to $+1$. Up to isotopy, we can suppose that there exist times $t_3 < t_1^j < \ldots < t_{2n_j+1}^j$ and $t_3 < t_1^k < \ldots t_{2n_k+1}^k$ such that $\phi'^{-1}(T_j) = \cup_{\ell=1}^{2n_j+1} \epsilon_\ell^j (S \times \{t_\ell^j\})$ and $\phi'^{-1}(T_k) = \cup_{\ell=1}^{2n_k+1} \epsilon_\ell^k (S \times \{t_\ell^k\})$, with ϵ_ℓ^j and ϵ_ℓ^k equal to $+1$ or -1, depending on the orientation of the component of $\phi'^{-1}(T_j)$ or $\phi'^{-1}(T_k)$ corresponding to the fiber $S \times \{t_\ell^j\}$ or $S \times \{t_\ell^k\}$. As $\sum_{\ell=1}^{n_j+1} \epsilon_\ell^j = +1$ and $\sum_{\ell=1}^{n_k+1} \epsilon_\ell^k = +1$, there exists ℓ and ℓ' such that $\epsilon_\ell^j = +1 = \epsilon_{\ell'}^k$. Suppose for example that $t_\ell^j < t_{\ell'}^k$. Then $\phi' : S \times [t_\ell^j, t_{\ell'}^k] \to V$ is a homotopy between the embedded surfaces T_j and T_k contained in the region in V bounded by S_3 and $S_{n'_i}$. As the embedded surface T_2 is not in this region, if we denote by Y the submanifold of C_i bounded by T_2 and $\partial_+ C_i$, the two embedded surfaces T_j and T_k are homotopic in the interior of Y.

By lemma 27, the surfaces T_j and T_k are incompressible in $C_i \setminus S_1$. As they are contained in the interior of Y and Y is included in the component of $C_i \setminus S_1$ containing T_j and T_k, the surfaces T_j and T_k are incompressible in Y. Thus, by a result of Waldhausen [Wal, Corollary 5.5 p. 76], they are in fact isotopic in Y. Therefore, T_j and T_k are parallel in C_i, for $3 < j < k \leq n'_i - 1$. Thus we have $m_i = n'_i - 4$ embedded surfaces $T_4, \ldots, T_{n'_i - 1}$ parallel in the compression body C_i, which ends the proof lemma 29 and also the proof of proposition 8. □

□

5. From patterns of fundamental domains to virtual fibration.

This part is dedicated to the proof of proposition 10, the "Pattern Lemma", which is based on [Mah, Lemma 4.12 p.2258]. This proof is much involved than the one of Lemma 4.12 in [Mah], which is too quick for our purpose since we need explicit bounds and precise constants. Therefore, we provide the necessary details.

Proof of proposition 10, the "Pattern Lemma".

Assume that there are m_i connected, separating, orientable, embedded and disjoint parallel surfaces in M_i, with diameter at most λ and at distance at least $r > 0$ from each other.

Let $\mathcal{D}$ be a Dirichlet fundamental domain for the manifold M in its universal cover $\widehat{M} \simeq \mathbb{H}^3$. The translates of $\mathcal{D}$ by the covering maps form a tiling of the universal cover $\widehat{M}$. This tiling descends to a tiling of the cover M_i by d_i copies of $\mathcal{D}$. Each of the m_i embedded and parallel surfaces $S_1, \ldots, S_{m_i}$ in M_i intersects some copies of $\mathcal{D}$. The union of copies intersected by one of those surfaces S_j is called a **pattern (of fundamental domains) for** S_j. As the surface is connected, a pattern is a connected 3-complex. We can suppose that each of the embedded surfaces intersects the 2-skeleton of the tiling transversally. More precisely, we can suppose that each surface does not meet the vertices of the fundamental polyhedra, that it intersects the edges in isolated points and it is transverse to the 2-dimensional faces of the polyhedra. Therefore, a pattern is a connected union of some copies of $\mathcal{D}$ glued along their 2-dimensional faces. Let us denote by D the diameter of $\mathcal{D}$, α the number of its 2-dimensional faces and β the number of faces of dimension zero, one and two.

For all $\ell \in \mathbb{N}$, we recall that $B(\ell)$ is an upper bound for the number of possibilities of patterns obtained by gluing together at most ℓ fundamental domains. Let $L = \lceil \frac{\beta\lambda}{\sigma} \rceil$, where σ is the

minimum over all pairs of non-adjacent faces of $\mathcal{D}$ of the distance between those faces. (As $\mathcal{D}$ is compact, σ is a well defined strictly positive real number.)

Let us denote by ν the greatest integer such that if $r/2D > 1$, $\frac{m_i}{\alpha^2 L^2 B(L)} \geq 2\nu + 3$ (which we will call condition (a)), or if $r/2D < 1$, such that $\left(\frac{r}{2D+1} m_i - 1\right) \frac{1}{\alpha^2 L^2 B(L)} \geq 2\nu + 3$ (called condition (b)). Let us suppose that $\nu \geq 2$.

• If two surfaces S_j and S_k are at distance strictly more than $2D$, the patterns of fundamental domains associated to S_j and S_k are necessarily disjoint, as the diameter of a fundamental domain is D.

If $r/2D > 1$, this is always the case for any pair of the m_i surfaces, and all the m_i patterns associated to the parallel surfaces are disjoint.

Otherwise, $r/2D \leq 1$. If we are in case (b), we have $\left(\frac{r}{2D+1} m_i - 1\right) \frac{1}{\alpha^2 L^2 B(L)} \geq 2\nu + 3$, so there are at least $\lfloor \frac{r}{2D+1} m_i \rfloor \geq \frac{r}{2D+1} m_i - 1$ surfaces which are separated from each other by a distance greater than or equal to $2D + 1 > 2D$. Therefore all the corresponding patterns of fundamental domains are disjoint.

• The surfaces are connected, their diameter is less than λ, and the minimal distance between two fundamental domains that do not share any face is at least σ. Therefore, a given surface cannot intersect more than $L = \lceil \frac{\beta\lambda}{\sigma} \rceil$ fundamental domains.

• If one of the two conditions (a) or (b) above is satisfied, at least $(2\nu + 3)\alpha^2 L^2$ of the initial surfaces have disjoint corresponding patterns. As those surfaces are parallel, they correspond to a same immersed surface S' in M. Choose an orientation for S'. By the covering map, this orientation gives an orientation of the surface S_j for each j. As the $(2\nu + 3)\alpha^2 L^2$ surfaces are parallel, at least $\lceil \frac{(2\nu+3)\alpha^2 L^2}{2} \rceil \geq (\nu\alpha^2 L^2 + 2)$ surfaces are coherently oriented, meaning that the orientation of S_j and S_k induced by the orientation of S' in M is the same in the product (in the product region bounded by S_j and S_k, the surface S_j is oriented by the inward normal vector whereas the surface S_k is oriented by the outward normal vector, or in the contrary the surface S_j is oriented outwards and the surface S_k oriented inwards). The orientations of the patterns corresponding to those $(\nu\alpha^2 L^2 + 2)$ surfaces are also the same, as their orientation locally coincide with the orientation of M near S'. We obtain then $(\nu\alpha^2 L^2 + 2)$ coherently oriented surfaces $S_0, \ldots, S_{\nu\alpha^2 L^2+1}$ such that there exists a product $S \times [0, \nu\alpha^2 L^2 + 1]$ that is embedded in the manifold M_i and in which the surface S_j coincides with the fiber $S \times \{j\}$ for all indices j between 0 and $(\nu\alpha^2 L^2 + 1)$.

Let P_j be the pattern of fundamental domains corresponding to the surface S_j. The patterns P_j are all disjoint, but homeomorphic to the same pattern P (and the gluings of fundamental domains are in one-to-one correspondence). More precisely, for each j, there exists a homeomorphism $\varphi_j : P_j \to P$ which preserves the gluings of fundamental domains.

Lemma 30. *The boundary of the pattern P contains at least two connected components.*

Proof of lemma 30.

We only have to show that for example P_1 has at least two boundary components. But as the surface S_1 is contained in the interior of the pattern P_1, $P_1 \cap (S \times [0,1]) \neq \emptyset$ and $P_1 \cap (S \times [1,2]) \neq \emptyset$. The pattern P_1 is disjoint to S_0 and S_2, so the product regions $S \times [0,1]$ and $S \times [1,2]$ are not contained in P_1. By connexity of $S \times [0,1]$ and $S \times [1,2]$, the boundary of the pattern P_1 has at least two components, one as a subset of $S \times (0,1)$ and the other one in $S \times (1,2)$. This proves lemma 30. □

The boundary of the pattern P has at least two components, and the number of its components is at most the number of 2-dimensional faces of a fundamental domain times the number of fundamental domains in the pattern, i.e. αL. Let $\partial P = T_1 \cup T_2 \cup \ldots \cup T_k$, where the immersed surfaces T_j are the components of the boundary of the pattern P, with $2 \leq k \leq \alpha L$.

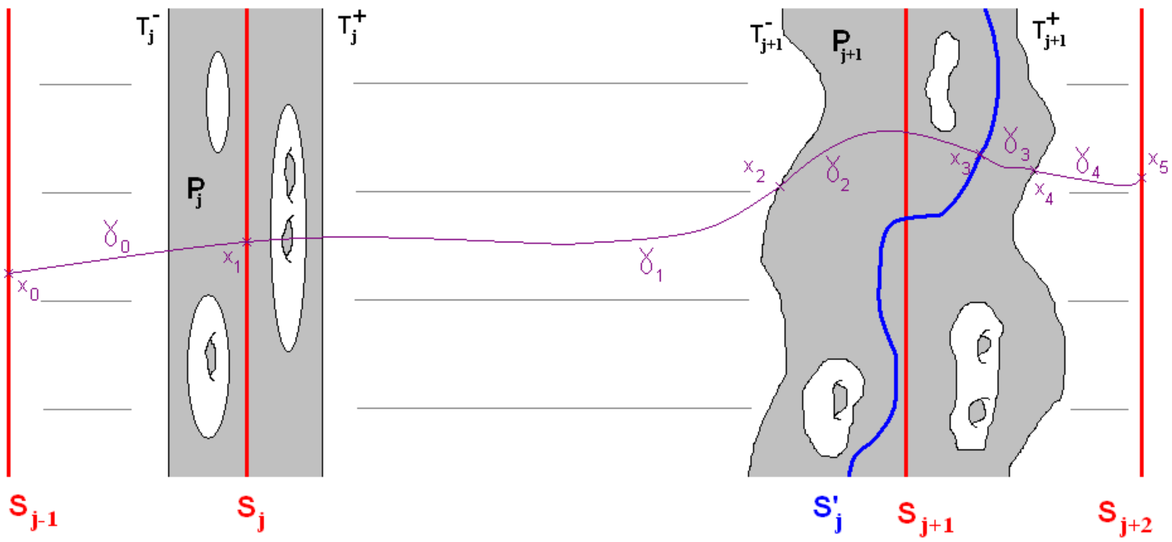
For every index j between 1 and $\nu\alpha^2 L^2$, the pattern P_j intersects $S \times (j-1, j)$ and $S \times (j, j+1)$. At least one component of the boundary of P_j is in the boundary of the component of $(S \times [j-1, j]) \setminus (S \times [j-1, j]) \cap P_j$ containing the fiber $S \times \{j-1\}$, which we will call a **"left" component of the boundary of the pattern** P_j. Similarly, at least one component of the boundary of P_j is in the boundary of the connected component of $(S \times [j, j+1]) \setminus (S \times [j, j+1]) \cap P_j$ containing

the fiber $S \times \{j+1\}$. We will call this component a **"right" component for the boundary of** P_j. For each index j between 1 and $\nu\alpha^2L^2$, take a left component (arbitrarily if there exist at least two such components) and a right component likewise for the pattern P_j. Those two components correspond to components T_j^- and T_j^+ of the boundary of P. As there are at most $\alpha L(\alpha L - 1) < \alpha^2 L^2$ pairs of such components of the boundary of P, there are at least ν surfaces $S_{j_1}, \dots, S_{j_\nu}$ with $1 \leq j_1 < \dots < j_\nu \leq \nu\alpha^2L^2$, for which the pairs of left and right components corresponding to the patterns P_{j_k} coincide. In the sequel, for brevity we will change the indices j and denote those surfaces by $S_1, \dots, S_\nu$.

Take two such surfaces S_j and S_{j+1} with $1 \leq j \leq \nu - 1$. The two patterns P_j and P_{j+1} are contained in the interior of the product $S \times [j-1, j+2]$. Let S'_j be the image of the surface S_j in the interior of the pattern P_{j+1} by the homeomorphism $\varphi_{j+1}^{-1} \circ \varphi_j$ between the patterns P_j and P_{j+1} : $S'_j = \varphi_{j+1}^{-1} \circ \varphi_j(S_j)$. This is an embedded surface in the product $S \times [j-1, j+2]$.

Lemma 31. *The homology class of S'_j in the product $S \times [j-1, j+2]$ is equal to the homology class of the fiber $[S] = [S_j] = [S_{j+1}]$.*

<u>Proof of lemma 31.</u>



By choice of the surfaces S_j and S_{j+1}, the left component T_j^- of the boundary of the pattern P_j and the left component T_{j+1}^- of the boundary of the pattern P_{j+1} have the same image in the pattern P : $\varphi_j(T_j^-) = \varphi_{j+1}(T_{j+1}^-)$, so $T_{j+1}^- = \varphi_{j+1}^{-1} \circ \varphi_j(T_j^-)$. By definition, T_{j+1}^- bounds the connected component of $(S \times [j, j+1]) \setminus (S \times [j, j+1]) \cap P_{j+1}$ containing the fiber S_j, and the component T_j^- of the boundary of the pattern P_j bounding the component of $(S \times [j-1, j]) \setminus (S \times [j-1, j]) \cap P_j$ containing the fiber S_{j-1}. As $P_j \cap (S \times [j-1, j])$ is connected, there exists a path γ'_2, properly embedded in $P_j \cap (S \times [j-1, j])$ and joining the component T_j^- to the surface S_j. The image by the homeomorphism $\varphi_{j+1}^{-1} \circ \varphi_j$ between the patterns P_j and P_{j+1} of the path γ'_2 is a path $\gamma_2 = \varphi_{j+1}^{-1} \circ \varphi_j(\gamma'_2)$ in P_{j+1} from the boundary component T_{j+1}^- to the surface S'_j. The interior of the path γ_2 is contained in the interior of the component of $P_{j+1} \setminus S'_j$ containing T_{j+1}^-. Let x_2 be the extremity of γ_2 belonging to the boundary component T_{j+1}^-, and x_3 the other one, on the surface S'_j.

Similarly, there exists a path γ_3 from x_3 to a point x_4 lying on the right component T_{j+1}^+ of the boundary of P_{j+1}, and such that its interior is contained in the interior of the component of $P_{j+1} \setminus S'_j$ containing T_{j+1}^+.

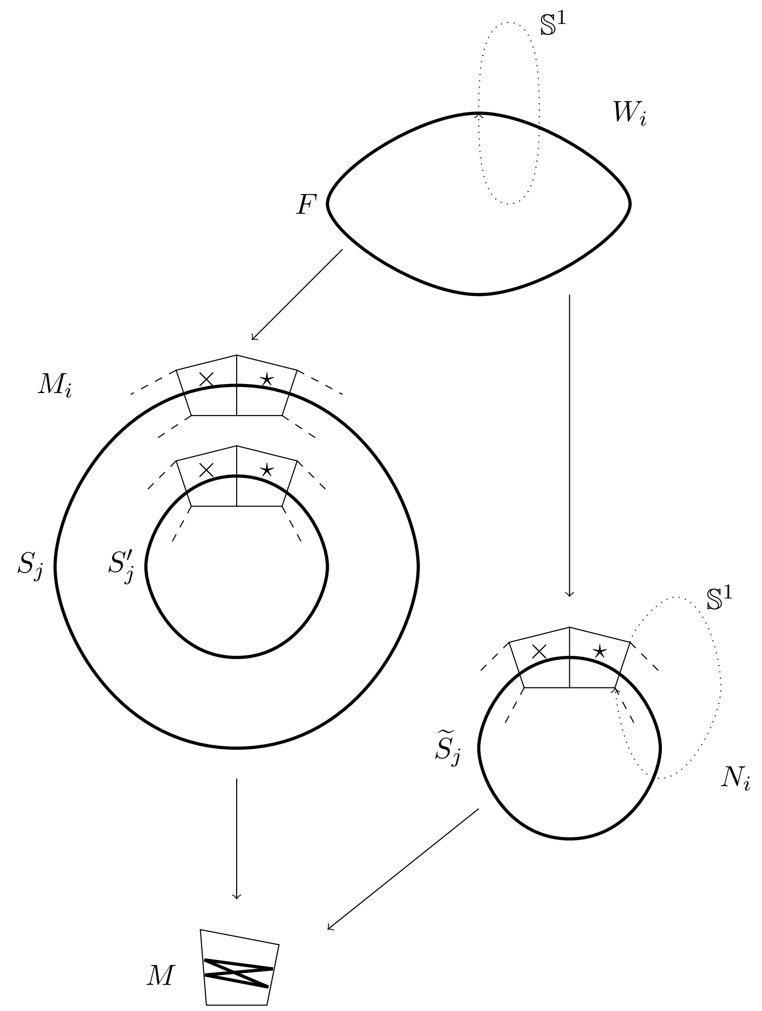
As T_{j+1}^- is in the boundary of the connected component of $(S \times [j, j+1]) \setminus P_{j+1} \cap (S \times [j, j+1])$ containing the fiber S_j, there exists a path γ_1 with its interior contained in the interior of this component, and joining a point x_1 of the fiber S_j to the point x_2 of T_{j+1}^-. Similarly, by choice of T_{j+1}^+, there exists a path γ_4 with interior contained in the interior of the component of

$(S\times[j+1,j+2])\setminus P_{j+1}\cap(S\times[j+1,j+2])$ containing the fiber S_{j+2} and linking the point x_4 of T^+_{j+1} to a point x_5 of S_{j+2}. Eventually, as the product $S\times[j-1,j]$ is connected, there exists a path γ_0 with interior contained in $S\times(j-1,j)$ joining the point x_1 of S_j to a point x_0 of S_{j-1}.

Let γ be the path obtained by concatenating the paths γ_0, γ_1, γ_2, γ_3 and γ_4. The path γ joins the point x_0 of S_{j-1} to the point x_5 of S_{j+2} and intersects the surface S'_j only once, at the point x_3. As the orientations of the patterns P_j and P_{j+1} coincide, the intersection number of γ with the surface S'_j is $+1$. So it is the same as the intersection number of γ with the fiber S. By Poincaré duality, as the homology group $H_2(S\times[j-1,j+2],\mathbb{Z})$ is of rank one, generated by the class of the fiber $[S]$, the class of the surface S'_j is equal to the class of the fiber in the homology of the product, showing lemma 31. □

As the surface S'_j is embedded in the product $S\times[j-1,j+2]$, by a result of Waldhausen [Wal] S'_j is parallel to the fiber S_j, possibly after performing a finite number of compressions on S'_j. But as the surface S'_j is homeomorphic to S_j, it is of the same genus as the fiber S_j, so in fact there is no compression. Therefore, those two surfaces bound a product in M_i.

We can cut the manifold M_i open along those two disjoint surfaces S_j and S'_j. We keep only the component corresponding to the product region between the two parallel surfaces, and we identify the two surfaces via the homeomorphism $\psi=(\varphi^{-1}_{j+1}\circ\varphi_j)_{|S_j}$ to obtain a manifold N_i fibering over the circle, with fiber $\widetilde{S_j}=(S_j\sim S'_j)$. The homeomorphism $\varphi^{-1}_{j+1}\circ\varphi_j$ identifies the "left" part of the pattern P_{j+1} with the "left" part of the pattern P_j, so we get a pattern $\widetilde{P_j}$ corresponding to $\widetilde{S_j}$ in N_i homeomorphic to the pattern P : the "left" part of this pattern corresponds to the left part of the pattern P_{j+1} via the homeomorphism φ_{j+1}, and the "right" part of the pattern corresponds to the right part of P_j via the homeomorphism φ_j. As those homeomorphisms preserve the gluings between the 2-dimensional faces of the fundamental domains, the gluings between the fundamental domains in the pattern $\widetilde{P_j}$ are the same as the gluings in the model pattern P. Therefore, we obtain a tiling of N_i by finitely many copies of fundamental domains homeomorphic to $\mathcal{D}$ and with matching gluings. Thus, N_i is a finite cover of the original manifold M, and N_i is fibered over the circle, with fiber $\widetilde{S_j}$.



The two covers M_i and N_i admit a common regular finite cover W_i, which fibers over the circle as it is a finite cover of N_i. A component of the preimage of S_j by the covering projection $W_i \to M_i$ is a fiber F for the fibration of W_i over the circle, as it is also a component the preimage of the fiber $\widetilde{S}_j$ in N_i corresponding to S_j in the mapping torus. As F is incompressible in W_i, the surface S_j embedded in M_i we started from is also incompressible. (If not, the preimage of a compression disc D for S_j would be a family of compression discs for F, which necessarily bound discs in F. As the homotopy class of the boundary of those discs is zero in $\pi_1(F)$, it is mapped to zero by the application $\pi_1(F) \to \pi_1(S_j)$ induced by the covering map. The disc D is then nulhomotopic in S_j, which provides a contradiction.) Thus the embedded surface S_j is a virtual fiber in M_i, and is incompressible.

Therefore, the m_i initial parallel surfaces are virtual fibers for the manifold M_i, which ends the proof of proposition 10. □

The m_i embedded surfaces are not quasi-Fuchsian, as they are virtual fibers.

6. The strong sub-logarithmic Heegaard gradient never vanishes.

In this section, we eventually prove theorem 3 and corollary 4.

Proof of theorem 3.

Let us assume that the assumption of theorem 1 is satisfied : there exists an infinite family $(M_i \to M)_{i\in\mathbb{N}}$ of finite covers of M such that

$$\lim_{i\to+\infty} \frac{\chi_-^h(M_i)}{\sqrt{\ln\ln d_i}} = 0. \tag{4}$$

We saw that we can then build $m_i = (\lfloor \frac{1}{g(C_i)+1} \lceil \frac{1}{2}(\frac{\delta_i}{18\epsilon K_i} - \frac{4K_i+2K_i'}{9K_i})\rceil \rfloor - 3)$ orientable, embedded and parallel surfaces in M_i. Furthermore, under the assumption (4), we know that for each i greater than or equal to some $i_0 \in \mathbb{N}$, we can start from M_i to construct a finite cover N_i of M which fibers over the circle, and whose fiber is diffeomorphic to the parallel surfaces in M_i. The m_i surfaces in M_i are then embedded virtual fibers, thus incompressible surfaces.

Therefore, for i greater than or equal to i_0, we have m_i disjoint embedded incompressible surfaces in M_i, with as in section 2 :

$$\begin{aligned} m_i &= \lfloor \frac{1}{g(C_i)+1} \lceil \frac{1}{2}(\frac{\delta_i}{18\epsilon K_i} - \frac{4}{9} - \frac{2K_i'}{9K_i})\rceil \rfloor - 4 \\ &\geq \frac{1}{\chi_-^h(M_i)+4}\left(\frac{\ln\left(\frac{d_i}{\chi_-^h(M_i)+2}\right) + \ln\left(\frac{2\mathrm{Vol}(M)}{\pi}\right)}{18\epsilon a\chi_-^h(M_i)+18\epsilon b} - \frac{4}{9} - \frac{4a'(2g(C_i)-2)}{9a(2g(C_i)-2)+9b}\right) - 5. \end{aligned}$$

Furthermore, any two such surfaces are at distance at least $\epsilon K_i \geq \epsilon(2a+b)$ from each other.

Let F be a strongly irreducible Heegaard surface for the manifold M_i with minimal genus : $|\chi(F)| = \chi_-^{sh}(M_i)$. According to theorem 5, and more precisely to the result (3) of Pitts and Rubinstein, we can suppose that, up to isotopy, the surface F is a minimal or a pseudo minimal surface. Therefore, by lemma 12, the ϵ-diameter of F is bounded from above by a linear function of the absolute value of its Euler characteristic : if $a' = 6(\frac{21}{4} + \frac{3}{4\pi} + \frac{3}{4\epsilon} + \frac{2}{\sinh(\epsilon/4)^2})$, then the ϵ-diameter of F is less than or equal to $a'\chi_-^{sh}(M_i)$. As the surface F is connected, its diameter is bounded from above by $2\epsilon a'\chi_-^{sh}(M_i)$. Thus, the surface F cannot intersect more than $\lceil \frac{\mathrm{diam}(F)}{\epsilon(2a+b)} \rceil \leq \lceil \frac{2\epsilon a'\chi_-^{sh}(M_i)}{\epsilon(2a+b)} \rceil$ of the former incompressible surfaces. So there are at least $m_i - \lceil \frac{2a'\chi_-^{sh}(M_i)}{2a+b} \rceil$ incompressible surfaces in M_i disjoint from the surface F. But if this number is strictly positive, it means that there exists at least one incompressible surface embedded in M_i lying in the complement of F. This surface is then an incompressible surface embedded in a handlebody, which provides a contradiction. Therefore, we always have :

$$m_i - \lceil \frac{2a'\chi_-^{sh}(M_i)}{2a+b} \rceil \leq 0,$$

which implies that

$$m_i - \frac{2a'\chi_-^{sh}(M_i)}{2a+b} - 1 \leq 0,$$

so

$$\chi_-^{sh}(M_i) \geq \frac{2a+b}{2a'}(m_i - 1). \tag{5}$$

Lemma 32. *There exists a positive integer P such that if $\chi_-^h(M_i) \geq P$, then*

$$\chi_-^{sh}(M_i) \geq \frac{2a+b}{2a'}\left(\frac{\ln(d_i)}{36\epsilon a\chi_-^h(M_i)^2} - 2\right).$$

From the other hand, there exists a constant $C > 0$ such that for every i such that $\chi_-^h(M_i) \leq P$ and d_i is large enough,

$$\chi_-^{sh}(M_i) \geq C\ln(d_i).$$

Proof of lemma 32.

To prove the first part of the lemma, we have :

$$m_i \geq \frac{1}{\chi_-^h(M_i)+4}\left(\frac{\ln\left(\frac{d_i}{\chi_-^h(M_i)+2}\right) + \ln\left(\frac{2\mathrm{Vol}(M)}{\pi}\right)}{18\epsilon a\chi_-^h(M_i)+18\epsilon b} - \frac{4}{9} - \frac{4a'(2g(C_i)-2)}{9a(2g(C_i)-2)+9b}\right) - 5.$$

As $\frac{4a'(2g(C_i)-2)}{9a(2g(C_i)-2)+9b} \sim_{g(C_i)\to+\infty} \frac{4a'}{9a}$, there exists a constant $Q \in \mathbb{N}$ such that if $g(C_i) \geq Q$, then $\frac{4a'(2g(C_i)-2)}{9a(2g(C_i)-2)+9b} \leq \frac{5a'}{9a}$. As for each i such that $g(C_i) < Q$, $\frac{4a'(2g(C_i)-2)}{9a(2g(C_i)-2)+9b} \leq \frac{4a'(2Q-2)}{18a+b}$, if we denote by $A = \max\left(\frac{5a'}{9a}, \frac{4a'(2Q-2)}{18a+b}\right)$, we have

$$m_i \geq \frac{1}{\chi_-^h(M_i)+4}\left(\frac{\ln\left(\frac{d_i}{\chi_-^h(M_i)+2}\right)+\ln\left(\frac{2\mathrm{Vol}(M)}{\pi}\right)}{18\epsilon a\chi_-^h(M_i)+18\epsilon b} - \frac{4}{9} - A\right) - 5.$$

Now,

$$\frac{1}{\chi_-^h(M_i)+4}\left(\frac{\ln\left(\frac{d_i}{\chi_-^h(M_i)+2}\right)+\ln\left(\frac{2\mathrm{Vol}(M)}{\pi}\right)}{18\epsilon a\chi_-^h(M_i)+18\epsilon b} - \frac{4}{9} - A\right) - 5 \sim_{\chi_-^h(M_i)\to+\infty} \frac{\ln\left(\frac{d_i}{\chi_-^h(M_i)}\right)}{18\epsilon a\chi_-^h(M_i)^2},$$

so for $\chi_-^h(M_i)$ large enough,

$$m_i \geq \frac{\ln\left(\frac{d_i}{\chi_-^h(M_i)}\right)}{36\epsilon a\chi_-^h(M_i)^2} = \frac{\ln(d_i)}{36\epsilon a\chi_-^h(M_i)^2} - \frac{\ln(\chi_-^h(M_i))}{36\epsilon a\chi_-^h(M_i)^2}.$$

If $\chi_-^h(M_i)$ is large enough, say greater than or equal to a constant $P \in \mathbb{N}$, $\frac{\ln(\chi_-^h(M_i))}{36\epsilon a\chi_-^h(M_i)^2} \leq 1$, and so

$$m_i \geq \frac{\ln(d_i)}{36\epsilon a\chi_-^h(M_i)^2} - 1.$$

We report then in inequality (5) : if $\chi_-^h(M_i) \geq P$,

$$\chi_-^{sh}(M_i) \geq \frac{2a+b}{2a'}\left(\frac{\ln(d_i)}{36\epsilon a\chi_-^h(M_i)^2} - 2\right).$$

To prove the second part of lemma 32, we write inequality (5) in full details :

$$\chi_-^{sh}(M_i) \geq \frac{2a+b}{2a'}\left(\frac{1}{\chi_-^h(M_i)+4}\left(\frac{\ln\left(\frac{d_i}{\chi_-^h(M_i)+2}\right)+\ln\left(\frac{2\mathrm{Vol}(M)}{\pi}\right)}{18\epsilon a\chi_-^h(M_i)+18\epsilon b} - \frac{4}{9} - \frac{8a'}{9a\chi_-^h(M_i)+9b}\right) - 6\right).$$

If $\chi_-^h(M_i) \leq P$, then

$$\chi_-^{sh}(M_i) \geq \frac{2a+b}{2a'}\left(\frac{1}{P+4}\left(\frac{\ln\left(\frac{d_i}{P+2}\right)+\ln\left(\frac{2\mathrm{Vol}(M)}{\pi}\right)}{18\epsilon aP+18\epsilon b} - \frac{4}{9} - \frac{8a'}{9aP+9b}\right) - 6\right).$$

Therefore, there exists a constant $C > 0$ such that for d_i large enough, $\chi_-^{sh}(M_i) \geq C\ln d_i$. This ends the proof of lemma 32. □

Taking the two cases of lemma 32 into account when i becomes large,

$$\lim_{i\to+\infty} \frac{\chi_-^{sh}(M_i)}{(\ln d_i)^\theta} = +\infty$$

for any $\theta \in (0,1)$, which proves theorem 3. □

Remark 4. *In fact, we have proven the following stronger version of theorem 3.*
If $\lim_{i\to+\infty} \frac{\chi_-^h(M_i)}{\sqrt{\ln\ln d_i}} = 0$, *then for any map* $f : \mathbb{N} \to \mathbb{N}$ *such that* $f(n) =_{n\to+\infty} o(\ln n)$, *we have* $\lim_{i\to+\infty} \frac{\chi_-^{sh}(M_i)}{f(d_i)} = +\infty$.

Proof of corollary 4.

Let M be a connected, orientable and closed hyperbolic 3-manifold, and $\{M_i \to M\}_{i\in\mathbb{N}}$ the countable family of finite covers of M. For each $i \in \mathbb{N}$, let us denote by $\epsilon_i = \frac{\chi_-^h(M_i)}{\sqrt{\ln\ln d_i}}$.

Let P be the positive integer given by lemma 32. Let

$$\begin{aligned} I &= \{i \in \mathbb{N} \,|\, \epsilon_i \geq 1\}, \\ J &= \{j \in \mathbb{N} \,|\, \epsilon_j < 1 \text{ and } \chi_-^h(M_j) \geq P\}, \text{ and} \\ K &= \{k \in \mathbb{N} \,|\, \epsilon_k < 1 \text{ and } \chi_-^h(M_k) < P\}. \end{aligned}$$

Those three subsets form a partition of $\mathbb{N}$.

For every $i \in I$,

$$\frac{\chi_-^{sh}(M_i)}{\sqrt{\ln\ln d_i}} \geq \frac{\chi_-^{sh}(M_i)}{\sqrt{\ln\ln d_i}} = \epsilon_i \geq 1 > 0.$$

For every $j \in J$, by lemma 25,

$$\chi_-^{sh}(M_i) \geq \frac{2a+b}{2a'}\left(\frac{\ln(d_i)}{36\epsilon a \chi_-^h(M_i)^2} - 2\right).$$

Hence,

$$\begin{aligned} \frac{\chi_-^{sh}(M_j)}{\sqrt{\ln\ln d_j}} &\geq \frac{2a+b}{2a'}\left(\frac{\ln(d_j)}{36\epsilon a \chi_-^h(M_j)^2} - 2\right) \\ &\geq \frac{2a+b}{2a'}\left(\frac{\ln(d_j)}{36\epsilon a \ln\ln d_j} - 2\right) \geq A_J > 0, \end{aligned}$$

where A_J is a strictly positive constant, independent on $j \in J$.

Again by lemma 32, there exists a constant $C > 0$ such that for every $k \in K$ with d_k large enough,

$$\frac{\chi_-^{sh}(M_k)}{\sqrt{\ln\ln d_k}} \geq C\frac{\ln d_k}{\sqrt{\ln\ln d_k}} \geq A_K > 0,$$

where A_K is independent on $k \in K$. We can choose A_K such that for every $k \in K$ such that if we do not have $\chi_-^{sh}(M_k) \geq C\ln d_k$, which are finitely many, the inequality $\frac{\chi_-^{sh}(M_k)}{\sqrt{\ln\ln d_k}} \geq A_K > 0$ still holds.

Let us denote by A the minimum of 1, A_J and A_K. As $\mathbb{N} = I \cup J \cup K$, we have for every $n \in \mathbb{N}$,

$$\frac{\chi_-^{sh}(M_n)}{\sqrt{\ln\ln d_n}} \geq A > 0,$$

so

$$\nabla_{log}^{sh}(M) = \inf_{n\in\mathbb{N}} \frac{\chi_-^{sh}(M_n)}{\sqrt{\ln\ln d_n}} \geq A > 0,$$

which ends the proof of corollary □

Références

[BCW] D. Bachman, D. Cooper, M. E. White, *Large embedded balls and Heegaard genus in negative curvature.* Algebr. Geom. Toplo. **4** (2004) 31–47.

[BO'N] R. L. Bishop, B. O'Neill, *Manifolds of negative curvature.* Trans. Amer. Math. Soc. **145** (1969) 1–49.

[BH] M. R. Bridson, A. Haefliger, *Metric spaces of non-positive curvature.* Grundlehren series **319**, Springer Verlag, Berlin (1999).

[CaGo] A. Casson, C.McA. Gordon, *Reducing Heegaard splittings.* Top. Appl. **27** (1987) 275–283.

[CoDL] T. H. Colding, C. de Lellis, *The min-max construction of minimal surfaces.* Surveys in Diff. Geom. **VIII**, Int. Press. Somerville, MA (2003) 75–107.

[FHS] M. Freedman, J. Hass, P. Scott, *Least area incompressible surfaces in 3-manifolds.* Invent. Math. **71** (1983) 609–642.

[Fro] C. Frohman, *The topological uniqueness of triply periodic minimal surfaces in* $\mathbb{R}^3$. J. Diff. Geom. **31** (1990) 277–283.

[FrHa] C. Frohman, J. Hass, *Unstable minimal surfaces and Heegaard splittings.* Invent. Math. **95** (1989) 529–540.

[Ga] D. Gabai, *Foliations and the topology of 3-manifolds.* J. Diff. Geom. **18** (1983) 445–503.

[Hak] W. Haken, *Some results on surfaces in 3-manifolds.* Studies in Modern Topology, Math. Assoc. Am. (1968) 34–98.

[Hem1] J. Hempel, *Residual finiteness for 3-manifolds.* Annals of Math. Studies **111** (1987) 379–396.

[Jac] W. Jaco, *Lectures on three-manifold topology.* Regional conference series in Mathematics **43**, Amer. Math. Soc. (1981).

[Lac2004] M. Lackenby, *The asymptotic behaviour of Heegaard genus.* Math. Res. Lett. **11** (2004) no. 2-3, 139–149.

[Lac2006] M. Lackenby, *Heegaard splittings, the virtually Haken conjecture and Property* (τ)*.* Invent. Math. **164** (2006) 317–359.

[Mah] J. Maher, *Heegaard gradient and virtual fibers.* Geom. and Top. **9** (2005) 2227–2259.

[MR] J. Maher, J. H. Rubinstein, *Period three actions on the three-sphere.* Geom. Topol. **7**, (2003) 329–397.

[MSY] W. Meeks III, L. Simon, S. Yau, *Embedded minimal surfaces, exotic spheres, and manifolds with positive Ricci curvature.* Ann. of Math. **116** (1982) 621–659.

[Pi] J. Pitts, *Existence and regularity of minimal surfaces on Riemannian manifolds.* Mathematical notes, **27**, Princeton University Press (1981).

[PiRu] J. Pitts, J.H. Rubinstein, *Existence of minimal surfaces of bounded topological type in three-manifolds.* Miniconference on Geometry and Partial Differential Equations (Camberra 1985) 163–176.

[Scha] M. Scharlemann, *Heegaard splittings of compact 3-manifolds.* Handbook of geometric topology, ed. by R. J. Daverman, R. B. Sher, chapter 18 (2002) 921–953.

[ST1] M. Scharlemann, A. Thompson, *Heegaard splittings of* $(surface) \times I$ *are standard.* Math. Ann. **295** (1993) 549–564.

[ST2] M. Scharlemann, A. Thompson, *Thin position for 3-manifolds.* AMS Contemp. Math. **164** (1994) 231–238.

[SY] R. Schoen, S. Yau, *Existence of incompressible surfaces and the topology of 3-manifolds with non-negative scalar curvature.* Ann. of Math. **119** (1979) 127–142.

[Sha] P. Shalen, *Hyperbolic volume, Heegaard genus, and ranks of groups.* Geometry and Topology Monographs **12** (2007) 335–349.

[Sou] J. Souto, *Geometry, Heegaard splittings and rank of the fundamental group of hyperbolic 3-manifolds.* Geometry and Topology Monographs, **12** (2007), avalable on the web : http ://www.math.uchicago.edu/∼juan/Haifa.pdf.

[Sta] J. Stallings, *On fibering certain 3-manifolds.* 1962 Topology of 3-manifolds and related topics, Proc. The Univ. of Georgia Institute (1961) 95–100.

[Wal] F. Waldhausen, *On irreducible 3-manifolds which are sufficiently large.* Ann. of Math. (2) **87** (1968) 56–88.

Claire RENARD
Université Paul Sabatier
Institut de Mathématiques de Toulouse
118 route de Narbonne
F-31062 Toulouse Cedex 9
claire.renard@math.univ-toulouse.fr